3.3 Equivalence Relations

Introduction to Equivalence Relations

An **equivalence relation** on a set is a way to formalize the idea of two elements being "related" to each other in a specific way. To define an equivalence relation, we need to specify three properties:

Definition of an Equivalence Relation

Let AAA be a set, and let RRR be a relation on AAA (i.e., a subset of A×AA \times AA×A). The relation RRR is called an equivalence relation if it satisfies the following three properties:

1. **Reflexivity:** For all a∈Aa \in Aa∈A, a R aa \, R \, aaRa holds.

   This means that every element is related to itself.

2. **Symmetry:** For all a,b∈Aa, b \in Aa,b∈A, if a R ba \, R \, baRb, then b R ab \, R \, abRa.

   This means if aaa is related to bbb, then bbb is related to aaa.

3. **Transitivity:** For all a,b,c∈Aa, b, c \in Aa,b,c∈A, if a R ba \, R \, baRb and b R cb \, R \, cbRc, then a R ca \, R \, caRc.

   This means that if aaa is related to bbb, and bbb is related to ccc, then aaa must be related to ccc.

Examples of Equivalence Relations

- **Equality on Numbers:** The relation === on the set of integers Z\mathbb{Z}Z is an equivalence relation since:
  - Reflexive: a=aa = aa=a for all a∈Za \in \mathbb{Z}a∈Z.
  - Symmetric: If a=ba = ba=b, then b=ab = ab=a.
  - Transitive: If a=ba = ba=b and b=cb = cb=c, then a=ca = ca=c.
- **Congruence Modulo nnn:** Define a≡b (mod n)a \equiv b \, (\text{mod} \, n)a≡b(modn) if nnn divides a−ba - ba−b. This is an equivalence relation on Z\mathbb{Z}Z:
  - Reflexive: a−a=0a - a = 0a−a=0, so a≡a (mod n)a \equiv a \, (\text{mod} \, n)a≡a(modn).
  - Symmetric: If a≡b (mod n)a \equiv b \, (\text{mod} \, n)a≡b(modn), then b≡a (mod n)b \equiv a \, (\text{mod} \, n)b≡a(modn).
  - Transitive: If a≡b (mod n)a \equiv b \, (\text{mod} \, n)a≡b(modn) and b≡c (mod n)b \equiv c \, (\text{mod} \, n)b≡c(modn), then a≡c (mod n)a \equiv c \, (\text{mod} \, n)a≡c(modn).
- **Equivalence of Rational Numbers by Same Value:** Define the relation ∼\sim∼ on Q\mathbb{Q}Q (the set of rational numbers) by a∼ba \sim ba∼b if and only if a=ba = ba=b. This is an equivalence relation since:
  - Reflexive: a=aa = aa=a.
  - Symmetric: If a=ba = ba=b, then b=ab = ab=a.
  - Transitive: If a=ba = ba=b and b=cb = cb=c, then a=ca = ca=c.

Equivalence Classes

An equivalence relation on a set AAA divides AAA into disjoint subsets, called **equivalence classes**. The equivalence class of an element a∈Aa \in Aa∈A, denoted [a][a][a], is the set of all elements in AAA that are related to aaa.

Formally, for a relation RRR on AAA, the equivalence class of an element aaa is defined as:

[a]={x∈A∣a R x}[a] = \{ x \in A \mid a \, R \, x \}[a]={x∈A∣aRx}

For example, if the relation is ∼\sim∼ on Z\mathbb{Z}Z defined by a∼ba \sim ba∼b if a=ba = ba=b, the equivalence class of 333 is [3]={3}[3] = \{ 3 \}[3]={3}.

Partitioning the Set

An equivalence relation on a set AAA naturally partitions AAA into disjoint equivalence classes. This means:

- Every element of AAA belongs to exactly one equivalence class.
- The equivalence classes are disjoint, i.e., if [a]∩[b]≠∅[a] \cap [b] \neq \emptyset[a]∩[b]□=∅, then [a]=[b][a] = [b][a]=[b].

This leads to the important result:

Theorem: *An equivalence relation on a set AAA partitions AAA into disjoint equivalence classes.*

Quotient Set (Set of Equivalence Classes)

The set of all equivalence classes of a set AAA under an equivalence relation RRR is called the **quotient set** or **set of equivalence classes** and is denoted by A/RA / RA/R. Formally:

A/R={[a]∣a∈A}A / R = \{ [a] \mid a \in A \}A/R={[a]∣a∈A}

Properties of Equivalence Classes

- **Uniqueness:** Each element of AAA belongs to one and only one equivalence class.
- **Disjointness:** If two equivalence classes [a][a][a] and [b][b][b] are not the same, then they are disjoint. That is, [a]∩[b]=∅[a] \cap [b] = \emptyset[a]∩[b]=∅ if a≠ba \neq ba□=b.

Problem Examples

Problem 1: Verify if ∼\sim∼ is an equivalence relation

Let A=ZA = \mathbb{Z}A=Z and define the relation ∼\sim∼ on AAA by a∼ba \sim ba∼b if and only if a−ba - ba−b is divisible by 3. Verify that ∼\sim∼ is an equivalence relation on Z\mathbb{Z}Z.

Solution:

- **Reflexive:** For any a∈Za \in \mathbb{Z}a∈Z, a−a=0a - a = 0a−a=0, which is divisible by 3. So a∼aa \sim aa∼a.
- **Symmetric:** If a∼ba \sim ba∼b, then a−ba - ba−b is divisible by 3. Since b−a=−(a−b)b - a = -(a - b)b−a=−(a−b), which is also divisible by 3, b∼ab \sim ab∼a.
- **Transitive:** If a∼ba \sim ba∼b and b∼cb \sim cb∼c, then a−ba - ba−b and b−cb - cb−c are divisible by 3. Therefore, (a−b)+(b−c)=a−c(a - b) + (b - c) = a - c(a−b)+(b−c)=a−c is divisible by 3, so a∼ca \sim ca∼c.

Thus, ∼\sim∼ is an equivalence relation on Z\mathbb{Z}Z.

Problem 2: Find the equivalence class of 222 under ∼\sim∼ from Problem 1.

Solution: The equivalence class of 222 under ∼\sim∼ is the set of all integers x∈Zx \in \mathbb{Z}x∈Z such that x−2x - 2x−2 is divisible by 3. This can be written as:

[2]={x∈Z∣x−2≡0 (mod 3)}={…,−4,−1,2,5,8,… }[2] = \{ x \in \mathbb{Z} \mid x - 2 \equiv 0 \, (\text{mod} \, 3) \} = \{ \dots, -4, -1, 2, 5, 8, \dots \}[2]={x∈Z∣x−2≡0(mod3)}={…,−4,−1,2,5,8,…}

In general, the equivalence class [2][2][2] consists of all integers that are congruent to 2 modulo 3.

Problem 3: Partition Z\mathbb{Z}Z using equivalence modulo 3.

Solution: The set of equivalence classes of Z\mathbb{Z}Z under congruence modulo 3 consists of the following three classes:

Z/∼={[0],[1],[2]}\mathbb{Z} / \sim = \{ [0], [1], [2] \}Z/∼={[0],[1],[2]}

Where:

- [0]={…,−3,0,3,6,… }[0] = \{ \dots, -3, 0, 3, 6, \dots \}[0]={…,−3,0,3,6,…},
- [1]={…,−2,1,4,7,… }[1] = \{ \dots, -2, 1, 4, 7, \dots \}[1]={…,−2,1,4,7,…},
- [2]={…,−4,−1,2,5,… }[2] = \{ \dots, -4, -1, 2, 5, \dots \}[2]={…,−4,−1,2,5,…}.

Applications of Equivalence Relations

Equivalence relations are used in many areas of mathematics and computer science, including:

- **Group theory**: Equivalence relations are closely tied to the concept of cosets.
- **Geometry**: Congruence relations define geometric equivalence.
- **Automata theory**: States of finite automata are often considered equivalent based on the language they recognize.