2.4 Quantifiers

Normal Forms

Elementary Product

A product of the statement variables and their negations in the formula is called Elementary Product.

The possible elementary products are

$P, Q, \neg P \wedge Q, \neg Q \wedge P, P \wedge \neg P, Q \wedge \neg Q, P \wedge \neg P \wedge Q$

Elementary Sum

A sum of the two statement variables and their negations is called Elementary Sum.

The possible elementary sums are

$P, Q, \neg P \vee Q, \neg Q \vee P, P \vee \neg P \vee Q, P \vee Q$

Disjunctive Normal Forms (DNF)

A statement formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula,

DNF = (Elementary product) ∨ (Elementary product) ∨ . . . ∨ (Elementary product)

Conjunctive Normal Forms (DNF)

A statement formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula,

DNF = (Elementary product) ∨ (Elementary product) ∨ . . . ∨ (Elementary product)

Remark:

(i) Note that DNF and CNF of given statement formula need not be unique.

(ii) In DNF and CNF, the number of variables in each term need not be same.

1. Obtain a disjunctive Normal form $P \wedge (P \rightarrow Q)$

Solution:

$P \wedge (P \rightarrow Q)$	Reason
$\Rightarrow P \wedge (\neg P \vee Q)$ $\Rightarrow (P \wedge \neg P) \vee (P \wedge Q)$	$(P \rightarrow Q \Rightarrow \neg P \vee Q)$ (Distributive law)

Since the given statement formula is written in terms of sum of elementary product.

DNF of $P \wedge (P \rightarrow Q)$ is $(P \wedge \neg P) \vee (P \wedge Q)$

2. Obtain DNF of $\neg(\boldsymbol{P} \vee \boldsymbol{Q}) \rightleftharpoons (\boldsymbol{P} \wedge \boldsymbol{Q})$

Solution:

$\neg(P \vee Q) \rightleftharpoons (P \wedge Q)$	Reason
$\Rightarrow [\neg(P \vee Q) \wedge (P \wedge Q)]$ $\vee\ [(P \vee Q) \wedge \neg(P \wedge Q)]$	$\big(R \rightleftharpoons S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)\big)$
$\Rightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$	(DeMorgan's law & Associative law)
$\Rightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee \big[P \wedge (\neg P \vee \neg Q) \vee \big(Q \wedge (\neg P \vee \neg Q)\big)\big]$	(Distributive law)
$\Rightarrow (P \wedge Q \wedge \neg P \wedge \neg Q) \vee (P \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee (Q \wedge \neg Q)$	(Distributive law)

Which is the required DNF.

Principal Normal Forms:

Min terms:

Let P and Q be 2 statement variables. Let us construct all possible formulas which consist of conjunction of P or its negation and conjunction of Q or its negation.

None of the formulas should contain both a variable and its negation. Using commutative law, if any two terms are equivalent choose any one of the term. Collect the remaining terms. They are called minterms.

For example, let P and Q be two variables, then the minterms are

$$P \wedge Q, P \wedge \neg Q, \neg P \wedge Q, \neg P \wedge \neg Q$$

Remark: 1

1. If there are "n" variables then the number of minterms is 2^n.

2. In elementary product a variable and its negation exist. But in minterms such things does not exist.

3. Let P, Q and R be 3 variables. The possible minterms are

1. $P \wedge Q \wedge R$

2. $P \wedge Q \wedge \neg R$

3. $P \wedge \neg Q \wedge R$

4. $\neg P \wedge Q \wedge R$

5. $\neg P \wedge \neg Q \wedge R$

6. $\neg P \wedge Q \wedge \neg R$

7. $P \wedge \neg Q \wedge \neg R$

8. $\neg P \wedge \neg Q \wedge \neg R$

Max terms:

Let P and Q be 2 statement variables. Let us construct all possible conjunction of disjunction P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law, if any two terms are equivalent choose any one of the term. Collect the remaining terms. They are called maxterms.

The possible maxterms with 2 variables are

$$P \vee Q, P \vee \neg Q, \neg P \vee Q, \neg P \vee \neg Q$$

Principal Disjunctive Normal Forms (PDNF)

For a given statement formula, an equivalent formula consisting of disjunction of minterms only is known as its Principal Disjunctive Normal Forms (PDNF)

PDNF = (minterms) ∨ (minterms) ∨ . . . ∨ (minterms)

Principal Conjunctive Normal Forms (PCNF)

For a given statement formula, an equivalent formula consisting of conjunction of maxterms only is known as its Principal Conjunctive Normal Forms (PCNF)

PCNF = (maxterms) ∧ (maxterms) ∧ . . . ∧ (maxterms)

Note:

1. PDNF is also called sum –of – products canonical form. PCNF is also called product – of – sums canonical form.

2. PDNF and PCNF of a given statement formula need not be unique.

PDNF and PCNF using Truth table

Using truth table, we can easily find PDNF and PCNF of given statement formulas.

Working rule to find PDNF:

1. Construct truth table for the given statement formula.

2. Choose each and every row in which the final column value is “TRUE”

3. In the selected row, if the truth value of each individual variable value is TRUE select that variable and truth value is FALSE then select the negation of that variable. In such a way collect all possible minterms.

4. Sum of all minterms gives the required PDNF.

Working rule to find PCNF:

1. Construct truth table for the given statement formula.

2. Choose each and every row in which the final column value is “FALSE”

3. In the selected row, if the truth value of each individual variable value is FALSE select that variable and truth value is TRUE then select the negation of that variable. In such a way collect all possible maxterms.

4. Product of all maxterms gives the required PCNF.

Problems under PDNF and PCNF using Truth table

1. Obtain PDNF of $P \rightarrow Q$

Solution:

P	Q	$P \rightarrow Q$	Min term
T	T	T	$P \wedge Q$
T	F	F	-
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

2. Obtain the PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solution:

P	Q	R	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$	Min term
T	T	T	T	F	T	T	$P \wedge Q \wedge R$
T	T	F	T	F	F	T	$P \wedge Q \wedge \neg R$
T	F	T	F	F	F	F	
T	F	F	F	F	F	F	
F	T	T	F	T	T	T	$\neg P \wedge Q \wedge R$
F	T	F	F	F	F	F	
F	F	T	F	T	F	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	F	F	F	F	

The PDNF is $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

3. Find the PCNF and PDNF of the proposition $\boldsymbol{P} \wedge (\boldsymbol{Q} \rightarrow \boldsymbol{R})$

Solution:

P	Q	R	$Q \rightarrow R$	$P \wedge (Q \rightarrow R)$	Min term	Max term
T	T	T	T	F		$P \vee Q \vee R$
T	T	F	T	F		$P \vee Q \vee \neg R$
T	F	T	F	F		$P \vee \neg Q \vee R$

T	F	F	T	F		$P \vee \neg Q \vee \neg R$
F	T	T	T	T	$P \wedge \neg Q \wedge \neg R$	
F	T	F	T	T	$P \wedge \neg Q \wedge R$	
F	F	T	F	F		$\neg P \vee \neg Q \vee R$
F	F	F	T	T	$P \wedge Q \wedge R$	

The PDNF is $(P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$

The PCNF is $(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge$

$(\neg P \vee \neg Q \vee R)$

4. Find the Principal Conjunctive normal form of $(\boldsymbol{P} \wedge \boldsymbol{A}) \wedge (\neg \boldsymbol{P} \wedge \boldsymbol{R})$

Solution:

P	Q	R	$\neg P$	$P \wedge Q$	$\neg P \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R)$	Min term
T	T	T	F	T	F	T	
T	T	F	F	T	F	T	
T	F	T	F	F	F	F	$\neg P \vee Q \vee \neg R$
T	F	F	F	F	F	F	$\neg P \vee Q \vee R$
F	T	T	T	F	T	T	
F	T	F	T	F	F	F	$P \vee \neg Q \vee R$

F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

The required PCNF is $(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$