2. 2. Propositional Equivalences

Logical Equivalences or Equivalence Rules

Laws	**Formulae**
Idempotent Laws	$p \wedge p \Leftrightarrow p, p \vee p \Leftrightarrow p$
Associative Laws	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
Commutative Laws	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee p \Leftrightarrow q \vee p$
DeMorgan's Laws	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Complement Laws	$p \wedge \neg p \Leftrightarrow F, p \vee \neg p \Leftrightarrow T$
Dominance Laws	$p \vee T \Leftrightarrow T, p \wedge F \Leftrightarrow F$
Identity Laws	$p \wedge T \Leftrightarrow p, p \vee F \Leftrightarrow p$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
Double Negation Laws	$\neg(\neg p) \Leftrightarrow p$
Contra Positive Laws	$p \to q \Leftrightarrow \neg q \to \neg p$

Conditional as Disjunction	$p \to q \Leftrightarrow \neg p \vee q$
Biconditional as Conditional	$p \to q \Leftrightarrow (p \to q) \wedge (q \to p)$
Exportations laws	$p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

1.Determine whether (¬Q ∧ (P →Q)) → ¬P is a tautology.

Solution:

(¬Q ∧ (P →Q)) → ¬P	Reason
$\Rightarrow (\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Rightarrow \neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Rightarrow (Q \vee (P \wedge \neg Q)) \vee \neg P$	(DeMorgan's law)
$\Rightarrow ((Q \vee P) \wedge (Q \vee \neg Q)) \vee \neg P$	(Distributive law)
$\Rightarrow ((Q \vee P) \wedge T) \vee \neg P$	$P \vee \neg P \Leftrightarrow T$
$\Rightarrow (Q \vee P) \vee \neg P$	$P \wedge T \Leftrightarrow P$
$\Rightarrow (Q \vee P \vee \neg P)$	(Associative law)
$\Rightarrow (Q \vee T)$	$P \vee \neg P \Leftrightarrow T$
$\Rightarrow T$	$P \vee T \Leftrightarrow T$

2. Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution:

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$	Reason
$\Rightarrow Q \vee (P \vee \neg P) \wedge \neg Q$ $\Rightarrow (Q \vee (P \vee \neg P)) \vee (Q \vee \neg Q)$ $\Rightarrow (Q \vee T) \wedge T$ $\Rightarrow T \wedge T$	(Distributive law) (Distributive law) $P \vee \neg P \Leftrightarrow T$ $P \vee T \Leftrightarrow P$

3. Show that the formula $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

Solution:

$(P \wedge Q) \rightarrow (P \vee Q)$	Reason
$\Rightarrow \neg(P \wedge Q) \vee (P \vee Q)$ $\Rightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ $\Rightarrow (P \vee \neg P) \vee (Q \vee \neg Q)$ $\Rightarrow T \vee T = T$	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$ (DeMorgan's law) (Associative law) (Negation law)

4. Show that the formula $\left(\neg P \wedge (\neg Q \wedge R)\right) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$	Reason
$\Rightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$ $\Rightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ $\Rightarrow [(P \vee \neg Q) \vee (Q \vee P)] \wedge R$ $\Rightarrow [\neg(P \vee Q) \vee (P \vee Q)] \wedge R$ $\Rightarrow T \wedge$ R $\Rightarrow$ R	(Distributive law) (Associative law) (Distributive law) (DeMorgan's law) $P \vee \neg P \Leftrightarrow T$ $P \wedge T \Leftrightarrow P$

Equivalence

Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ or $A \leftrightarrows B$ is a tautology. It is denoted by the symbol $A \Leftrightarrow B$ which is read as "A is equivalent to B."

Remark:

To prove two statement formulas A and B are equivalent, we can use any one of the following method:

(i) using Truth Table, we show that truth values of both statements formulas A and B are same for each 2^n combinations.

(ii) Assume A. By applying various equivalence rules try to derive B and vice versa.

(iii) Prove $A \leftrightharpoons B$ is a tautology.

1. Show that $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

Solution:

$\neg(P \vee (\neg P \wedge Q))$	Reason
$\Leftrightarrow \neg P \wedge (\neg(\neg P \wedge \neg Q))$	(DeMorgan's law)
$\Leftrightarrow \neg P \wedge [\neg(\neg P) \vee \neg Q]$	(DeMorgan's law)
$\Leftrightarrow \neg P \wedge (P \vee \neg Q)$	(Double Negation law)
$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$	(Distributive law)
$\Leftrightarrow F \vee (\neg P \wedge \neg Q)$	$\neg P \wedge P \Leftrightarrow F$
$\Leftrightarrow (\neg P \wedge \neg Q) \vee F$	(Commutative law)
$\Leftrightarrow \neg P \wedge \neg Q$	(identity law)

Hence $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

2. Prove that $P \rightarrow Q \Leftrightarrow P \rightarrow (P \wedge Q)$

Solution:

$P \rightarrow (P \wedge Q)$	Reason

$\Leftrightarrow \neg P \vee (P \wedge Q)$	(Conditional as disjunction)
$\Leftrightarrow (\neg P \vee P) \wedge (\neg P \wedge Q)$	(Distributive law)
$\Leftrightarrow T \wedge (\neg P \wedge Q)$	$\neg P \wedge P \Leftrightarrow F$
$\Leftrightarrow \neg P \wedge Q$	(Identity law)
$\Leftrightarrow P \rightarrow Q$	(Conditional as disjunction)

3. Prove that $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$

Solution:

$(P \rightarrow R) \wedge (Q \rightarrow R)$	Reason
$\Leftrightarrow (\neg P \wedge R) \wedge (\neg Q \wedge R)$	(Conditional as disjunction)
$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$	(Distributive law)
$\Leftrightarrow \neg(P \vee Q) \vee R$	(DeMorgan's law)
$\Leftrightarrow (P \vee Q) \rightarrow R$	(Conditional as disjunction)

4. Prove that $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Solution:

$P \rightarrow (Q \rightarrow R)$	Reason
$\Leftrightarrow \neg P \vee (Q \rightarrow R)$	(Conditional as disjunction)
$\Leftrightarrow \neg P \vee (\neg Q \vee R)$	(Conditional as disjunction)
$\Leftrightarrow \neg(\neg P \vee \neg Q) \vee R$	(Associative law)
$\Leftrightarrow \neg(P \wedge Q) \vee R$	(DeMorgan's law)
$\Leftrightarrow (P \wedge Q) \rightarrow R$	(Conditional as disjunction)