1.5 Cardinality of Sets

Introduction to Cardinality

Cardinality refers to the "size" or "number of elements" in a set. More formally, the cardinality of a set AAA is denoted as ∣A∣|A|∣A∣ or #A\#A#A. Cardinality can apply to both finite and infinite sets, though the treatment differs in each case.

- **Finite Sets:** The cardinality is simply the number of elements in the set.
- **Infinite Sets:** Cardinality involves comparing the "size" of sets using concepts such as bijections (one-to-one correspondences) and the concept of infinity.

Cardinality of Finite Sets

If a set is finite, its cardinality is simply the number of distinct elements in the set. For example:

- A={1,2,3}A = \{1, 2, 3\}A={1,2,3}, so ∣A∣=3|A| = 3∣A∣=3.
- B={}B = \{\}B={} (the empty set), so ∣B∣=0|B| = 0∣B∣=0.

Cardinality of Infinite Sets

Infinite sets are more complex because they don't have a finite number of elements. However, we can still talk about their cardinality by comparing them to one another. Two sets are said to have the same cardinality if there exists a bijection (a one-to-one correspondence) between them.

Countable vs. Uncountable Sets

- **Countably Infinite Sets**: A set is countably infinite if its elements can be placed in one-to-one correspondence with the natural numbers N\mathbb{N}N. Examples of countably infinite sets include:
  - N={1,2,3,4,… }\mathbb{N} = \{1, 2, 3, 4, \dots\}N={1,2,3,4,…} (the set of natural numbers).
  - Z={…,−2,−1,0,1,2,… }\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}Z={…,−2,−1,0,1,2,…} (the set of integers).
  - Q\mathbb{Q}Q (the set of rational numbers).

  The cardinality of any countably infinite set is denoted ℵ0\aleph_0ℵ0 (aleph-null or aleph-zero), which is the smallest level of infinity.

- **Uncountably Infinite Sets**: A set is uncountably infinite if its elements cannot be placed in one-to-one correspondence with the natural numbers. The classic example is the set of real numbers R\mathbb{R}R. The cardinality of R\mathbb{R}R is strictly greater than ℵ0\aleph_0ℵ0 and is denoted c\mathfrak{c}c (the cardinality of the continuum).

Comparing Cardinalities

To compare the cardinalities of two sets, we look for a bijection (a one-to-one correspondence) between them.

- **Two sets have the same cardinality** if there exists a bijection between them.
- **One set has greater cardinality** than another if no bijection can be formed between the two sets.

Cantor's Diagonal Argument (Uncountability of $\mathbb{R}$)

One of the most famous proofs in set theory is Cantor's diagonalization argument, which shows that the real numbers $\mathbb{R}$ are uncountable. The argument works by assuming that the real numbers between 0 and 1 can be listed in a sequence, and then constructing a new real number that differs from each number in the list at least in one decimal place, leading to a contradiction. Thus, the set of real numbers cannot be counted by natural numbers.

Operations on Cardinalities

We can perform various operations on sets, such as unions, intersections, and power sets, and analyze their effects on cardinality.

- **Union of Sets**: If A and B are sets, then $|A \cup B|$ is given by the principle of inclusion-exclusion:

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

  This formula generalizes for more than two sets.

- **Power Set**: The power set of a set A, denoted $P(A)$, is the set of all subsets of A. If $|A| = n$ is finite, then $|P(A)| = 2^n$. If A is infinite, the cardinality of $P(A)$ is strictly greater than the cardinality of A. For example, if $|A| = \aleph_0$, then $|P(A)| = 2^{\aleph_0} = \mathfrak{c}$.

Cardinality of the Continuum and Beyond

There are infinitely many different cardinalities of infinite sets. The continuum hypothesis deals with the size of the real numbers relative to the set of natural numbers. It asks whether there is a cardinality strictly between $\aleph_0$ and $\mathfrak{c}$. This question is independent of the standard axioms of set theory (ZFC) — it can neither be proven nor disproven from those axioms.

- The cardinality of the continuum $\mathfrak{c}$ is sometimes identified with $2^{\aleph_0}$, the cardinality of the set of real numbers.

Applications of Cardinality

- **Set Theory and Foundations of Mathematics**: Cardinality plays a central role in the study of infinite sets and helps define the structure of the continuum in mathematics.
- **Mathematical Logic**: Cardinality is used to classify different types of infinite sets in logic and to understand the structure of mathematical models.
- **Computational Complexity**: The cardinality of sets is sometimes used to discuss the size of problems in computational theory, especially when dealing with infinite objects or languages.

Basic Cardinality

Problem:
Let A={1,2,3,4,5}A = \{ 1, 2, 3, 4, 5 \}A={1,2,3,4,5}. What is the cardinality of set AAA?

Solution: The set AAA contains the elements 1,2,3,4,51, 2, 3, 4, 51,2,3,4,5. So, the cardinality of AAA, denoted by |A||A||A|, is simply the number of elements in the set.

Thus, the cardinality of AAA is:

|A|=5|A| = 5|A|=5

Cardinality of an Empty Set

Problem:
What is the cardinality of the empty set ∅\emptyset∅?

Solution: The empty set, denoted by ∅\emptyset∅, has no elements. Therefore, its cardinality is:

|∅|=0|\emptyset| = 0|∅|=0

Cardinality of a Set with Repeated Elements

Problem:
Let B={2,3,3,4,4,4,5}B = \{ 2, 3, 3, 4, 4, 4, 5 \}B={2,3,3,4,4,4,5}. What is the cardinality of set BBB?

Solution: The set BBB contains repeated elements. However, in a set, repeated elements are not counted more than once. Thus, the set can be rewritten as:

B={2,3,4,5}B = \{ 2, 3, 4, 5 \}B={2,3,4,5}

Now, counting the distinct elements, we see that there are 4 elements. Therefore, the cardinality of BBB is:

|B|=4|B| = 4|B|=4

Cardinality of a Power Set

Problem:
Let C={a,b}C = \{ a, b \}C={a,b}. What is the cardinality of the power set of CCC?

Solution: The **power set** of a set CCC is the set of all subsets of CCC, including the empty set and CCC itself.

The set C={a,b}C = \{ a, b \}C={a,b} has 2 elements. The number of subsets of a set with nnn elements is 2n2^n2n. Thus, for CCC, the number of subsets is:

22=42^2 = 422=4

The power set of CCC is:

P(C)={∅,{a},{b},{a,b}}\mathcal{P}(C) = \{\emptyset, \{a\}, \{b\}, \{a, b\} \}P(C)={∅,{a},{b},{a,b}}

Thus, the cardinality of the power set of CCC is:

|P(C)|=4|\mathcal{P}(C)| = 4|P(C)|=4

Cardinality of a Union of Two Sets

Problem:
Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. What is the cardinality of the union A∪BA \cup BA∪B?

Solution: The **union** of two sets AAA and BBB is the set of all elements that are in either AAA, BBB, or both. We combine the elements of both sets and eliminate duplicates.

A∪B={1,2,3,4,5}A \cup B = \{ 1, 2, 3, 4, 5 \}A∪B={1,2,3,4,5}

There are 5 distinct elements in A∪BA \cup BA∪B, so the cardinality of the union is:

|A∪B|=5|A \cup B| = 5|A∪B|=5

Cardinality of an Intersection of Two Sets

Problem:
Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. What is the cardinality of the intersection A∩BA \cap BA∩B?

Solution: The **intersection** of two sets AAA and BBB is the set of all elements that are common to both AAA and BBB.

A∩B={3}A \cap B = \{ 3 \}A∩B={3}

There is only 1 element in the intersection, so the cardinality of A∩BA \cap BA∩B is:

|A∩B|=1|A \cap B| = 1|A∩B|=1

Cardinality of a Difference of Two Sets

Problem:
Let A={1,2,3,4}A = \{ 1, 2, 3, 4 \}A={1,2,3,4} and B={3,4,5,6}B = \{ 3, 4, 5, 6 \}B={3,4,5,6}. What is the cardinality of the difference A−BA - BA−B?

Solution: The **difference** A−BA - BA−B is the set of elements that are in AAA but not in BBB.

A−B={1,2}A - B = \{ 1, 2 \}A−B={1,2}

Thus, the cardinality of A−BA - BA−B is:

|A−B|=2|A - B| = 2|A−B|=2

Cardinality of a Cartesian Product

Problem:
Let X={1,2}X = \{ 1, 2 \}X={1,2} and Y={a,b,c}Y = \{ a, b, c \}Y={a,b,c}. What is the cardinality of the Cartesian product X×YX \times YX×Y?

Solution: The **Cartesian product** X×YX \times YX×Y is the set of ordered pairs (x,y)(x, y)(x,y) where x∈Xx \in Xx∈X and y∈Yy \in Yy∈Y.

The number of elements in the Cartesian product is the product of the number of elements in XXX and the number of elements in YYY.

Since |X|=2|X| = 2|X|=2 and |Y|=3|Y| = 3|Y|=3, the cardinality of X×YX \times YX×Y is:

|X×Y|=2×3=6|X \times Y| = 2 \times 3 = 6|X×Y|=2×3=6

The Cartesian product X×YX \times YX×Y is:

X×Y={(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)}X \times Y = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}X×Y={(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)}

Thus, the cardinality of X×YX \times YX×Y is 6.