3.4 Partial Orderings

Partial Orderings

A **partial ordering** on a set AAA is a relation ≤\leq≤ on AAA that is:

1. **Reflexive**: For all a∈Aa \in Aa∈A, a≤aa \leq aa≤a.
2. **Antisymmetric**: For all a,b∈Aa, b \in Aa,b∈A, if a≤ba \leq ba≤b and b≤ab \leq ab≤a, then a=ba = ba=b.
3. **Transitive**: For all a,b,c∈Aa, b, c \in Aa,b,c∈A, if a≤ba \leq ba≤b and b≤cb \leq cb≤c, then a≤ca \leq ca≤c.

In a partially ordered set (poset), not every pair of elements must be comparable. That is, for some pairs aaa and bbb, neither a≤ba \leq ba≤b nor b≤ab \leq ab≤a may hold.

Hasse Diagrams

A **Hasse diagram** is a graphical representation of a partially ordered set (poset). In this diagram:

- Elements of the set are represented as vertices.
- There is an edge between two elements aaa and bbb if a≤ba \leq ba≤b and there is no element ccc such that a<c<ba < c < ba<c<b.
- The diagram is drawn such that higher elements are placed above lower elements, making the ordering structure visually apparent.

Example:

For the poset {a,b,c}\{a, b, c\}{a,b,c} with relations a≤ba \leq ba≤b and a≤ca \leq ca≤c, the Hasse diagram would have aaa at the bottom, and bbb and ccc above aaa, with edges between aaa and bbb, and between aaa and ccc.

Total Orderings

A **total ordering** (or **linear ordering**) is a special case of a partial ordering where every pair of elements is comparable. That is, for any two elements a,b∈Aa, b \in Aa,b∈A, either a≤ba \leq ba≤b or b≤ab \leq ab≤a.

Formally, a total ordering is a relation ≤\leq≤ on a set AAA that satisfies the following properties:

- **Reflexive**
- **Antisymmetric**
- **Transitive**
- **Total**: For every pair a,b∈Aa, b \in Aa,b∈A, either a≤ba \leq ba≤b or b≤ab \leq ab≤a (i.e., the relation is connected).

Equivalence Relations

An **equivalence relation** on a set AAA is a relation ∼\sim∼ that satisfies the following three properties:

- **Reflexive**: a∼aa \sim aa∼a for all a∈Aa \in Aa∈A.
- **Symmetric**: If a∼ba \sim ba∼b, then b∼ab \sim ab∼a.
- **Transitive**: If a∼ba \sim ba∼b and b∼cb \sim cb∼c, then a∼ca \sim ca∼c.

Equivalence relations divide a set into **equivalence classes**, which are subsets of AAA where all elements in each class are equivalent to each other.

Common Examples of Partial Orderings

1. **Subset Ordering**: For a set SSS, the relation ⊆\subseteq⊆ (subset) is a partial order. If A⊆BA \subseteq BA⊆B, then AAA is related to BBB. This relation is reflexive, antisymmetric, and transitive.
2. **Divisibility Ordering**: For the set of natural numbers N\mathbb{N}N, define the relation a≤ba \leq ba≤b if aaa divides bbb. This is a partial order.
3. **Task Scheduling**: Consider a set of tasks where task AAA must be completed before task BBB. The "before" relation defines a partial order.

Maximal and Minimal Elements

In a partially ordered set:

- **Maximal Element**: An element aaa is maximal if there is no element bbb such that a<ba < ba<b.
- **Minimal Element**: An element aaa is minimal if there is no element bbb such that b<ab < ab<a.
- **Greatest Element**: An element ggg is the greatest element if g≥ag \geq ag≥a for all a∈Aa \in Aa∈A.
- **Least Element**: An element lll is the least element if l≤al \leq al≤a for all a∈Aa \in Aa∈A.

Lattices

A **lattice** is a poset where every pair of elements has both a **supremum** (least upper bound) and an **infimum** (greatest lower bound). For two elements aaa and bbb in a lattice:

- The **supremum** of aaa and bbb, denoted a∨ba \vee ba∨b, is the least element greater than or equal to both aaa and bbb.
- The **infimum** of aaa and bbb, denoted a∧ba \wedge ba∧b, is the greatest element less than or equal to both aaa and bbb.

Lattices can be classified as **distributive** or **non-distributive** based on whether the operations ∨\vee∨ and ∧\wedge∧ distribute over each other.

Partial orderings are a fundamental concept in mathematics and computer science, used to describe a set with a relation that is reflexive, antisymmetric, and transitive. Partial orderings are important for understanding how elements in a set are related, even when not all elements are comparable.

Let me break down the key properties of a partial ordering:

1. **Reflexive**: Every element is related to itself.
   - ∀x∈S,x≤x\forall x \in S, x \leq x∀x∈S,x≤x
2. **Antisymmetric**: If two elements are related in both directions, they must be equal.
   - ∀x,y∈S,(x≤y and y≤x)⇒x=y\forall x, y \in S, (x \leq y \text{ and } y \leq x) \Rightarrow x = y∀x,y∈S,(x≤y and y≤x)⇒x=y
3. **Transitive**: If an element is related to a second, and the second is related to a third, then the first is related to the third.
   - ∀x,y,z∈S,(x≤y and y≤z)⇒x≤z\forall x, y, z \in S, (x \leq y \text{ and } y \leq z) \Rightarrow x \leq z∀x,y,z∈S,(x≤y and y≤z)⇒x≤z

Partial orderings are often represented using a **Hasse diagram**, where elements are shown as nodes, and edges represent the ordering between elements.

Some Types of Partial Orders

- **Total Order**: A special case of partial order where every pair of elements is comparable. That is, for any two elements xxx and yyy, either x≤yx \leq yx≤y or y≤xy \leq xy≤x.
- **Strict Partial Order**: A partial order where the reflexive property is not included (i.e., x≤xx \leq xx≤x does not hold).

Now, I will provide a few **problems** involving partial orderings, along with their solutions:

Problem 1: Verify if the Relation is a Partial Order

Let S={1,2,3,4}S = \{ 1, 2, 3, 4 \}S={1,2,3,4}, and let the relation ≤\leq≤ on SSS be defined by:

R={(1,1),(1,2),(2,2),(2,3),(3,3),(1,3),(4,4)}R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (1, 3), (4, 4) \}R={(1,1),(1,2),(2,2),(2,3),(3,3),(1,3),(4,4)}

Check if this relation is a partial order.

Solution:

To verify that RRR is a partial order, we need to check the three properties:

- **Reflexivity**: For each element x∈Sx \in Sx∈S, we need (x,x)(x, x)(x,x) to be in RRR.
    - For 111, (1,1)(1, 1)(1,1) is in RRR.
    - For 222, (2,2)(2, 2)(2,2) is in RRR.
    - For 333, (3,3)(3, 3)(3,3) is in RRR.
    - For 444, (4,4)(4, 4)(4,4) is in RRR.
    - Thus, RRR is **reflexive**.
- **Antisymmetry**: We need to check if (x,y)∈R(x, y) \in R(x,y)∈R and (y,x)∈R(y, x) \in R(y,x)∈R imply x=yx = yx=y.
    - There are no pairs (x,y)(x, y)(x,y) and (y,x)(y, x)(y,x) in RRR where x≠yx \neq yx□=y.
    - Thus, RRR is **antisymmetric**.
- **Transitivity**: We need to check if whenever (x,y)∈R(x, y) \in R(x,y)∈R and (y,z)∈R(y, z) \in R(y,z)∈R, then (x,z)(x, z)(x,z) must also be in RRR.
    - From (1,2)(1, 2)(1,2) and (2,3)(2, 3)(2,3), we must have (1,3)(1, 3)(1,3), which is in RRR.
    - All other combinations hold transitivity as well.
    - Thus, RRR is **transitive**.

Since all three properties are satisfied, RRR is a **partial order**.

Problem 2: Find the Hasse Diagram for a Partial Order

Consider the set S={a,b,c,d,e}S = \{a, b, c, d, e\}S={a,b,c,d,e} with the following partial order ≤\leq≤:

≤={(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(b,e),(c,e),(d,e)}\leq = \{ (a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (b, e), (c, e), (d, e) \}≤={(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(b,e),(c,e),(d,e)}

Draw the Hasse diagram for this partial order.

Solution:

1. **Start by plotting the elements**: The set S={a,b,c,d,e}S = \{a, b, c, d, e\}S={a,b,c,d,e} contains the elements a,b,c,d,ea, b, c, d, ea,b,c,d,e.
2. **Use the relation** to figure out which elements are connected:
    - a≤ba \leq ba≤b, a≤ca \leq ca≤c, and a≤da \leq da≤d, so aaa is the smallest element in the diagram.
    - b,c,db, c, db,c,d are all connected to eee, which is the largest element.
    - bbb is not comparable to ccc or ddd, so they stand as separate nodes.

3. **Simplify the diagram**:
   - We only draw the edges between elements where the relation is directly defined and remove transitive edges (i.e., if a≤ba \leq ba≤b and b≤eb \leq eb≤e, we don't need to show a≤ea \leq ea≤e).

Here is the **Hasse diagram**:

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css
Copy code
       e
     / |  \
    /  |   \
   b   c    d
   |
   a
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Problem 3: Find the Maximum and Minimum Elements

Let S={1,2,3,4,5}S = \{1, 2, 3, 4, 5\}S={1,2,3,4,5}, and define the partial order ≤\leq≤ by the divisibility relation. That is, x≤yx \leq yx≤y if and only if xxx divides yyy.

Find the maximum and minimum elements in this partial order.

Solution:

In this partial order, the elements are compared based on divisibility. Let's consider the divisibility relations:

- 111 divides all other elements, so 111 is the **minimum** element.
- 555 is divisible by itself, and no other element divides 555 except 111, so 555 is the **maximum** element.

Thus, the **minimum** element is 111 and the **maximum** element is 555.

Problem 4: Check if the Given Set is Totally Ordered

Consider the set S={2,4,6,8}S = \{ 2, 4, 6, 8 \}S={2,4,6,8} with the relation ≤\leq≤ defined by divisibility (i.e., x≤yx \leq yx≤y if xxx divides yyy).

Check if this set, with this relation, forms a **total order**.

Solution:

A set forms a **total order** if for every pair of elements x,y∈Sx, y \in Sx,y∈S, either x≤yx \leq yx≤y or y≤xy \leq xy≤x holds.

Let’s check the divisibility for every pair:

- 2≤42 \leq 42≤4, 2≤62 \leq 62≤6, and 2≤82 \leq 82≤8, since 222 divides all other elements.
- 4≤84 \leq 84≤8, since 444 divides 888.
- 666 does not divide 444, and 444 does not divide 666.

Since 444 and 666 are not comparable, the set is **not totally ordered**.