UNIT I

Mechanics

1.3 Theorems of Moment of Inertia (MI)

There are two important theorems which help to find the moment of inertia of a body about some other axis if moment of inertia about any symmetrical axis of the body is given. These are called theorems of parallel and perpendicular axes.

They are

1. Parallel axes theorem and
2. Perpendicular axes theorem

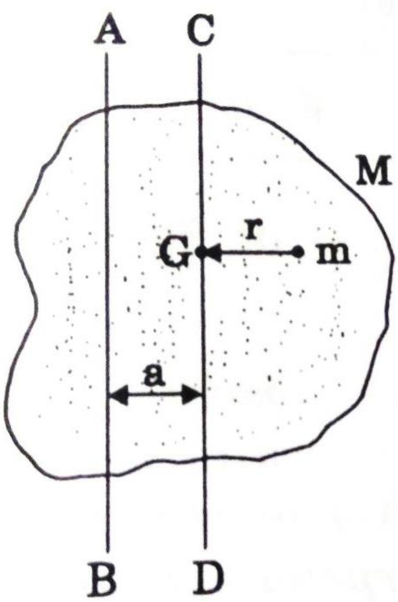
1. Theorem (Principle) of Parallel axes

Statement

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of gravity of the body and the product of its mass of the body with the square of the distance between the two axes.

Explanation

Let G be the centre of gravity of a rigidity of a rigid body of mass M. Let AB be an axis parallel to axis CD.



Let a be the distance between the axes.

If I and I_G are the moment of inertia of the body about the axes AB and CD respectively, then by the theorem of parallel axes,

$$I = I_G + Ma^2$$

Proof

Consider a particle of mass m at a distance r from CD.

M.I. of this particle about the axis CD = mr^2

$\therefore$ M.I. of the whole body about CD, $I_G = \Sigma\, mr^2$

M.I. of the particle about the axis AB = $m(r + a)^2$

$\therefore$ M.I. of the whole body about AB, $I = \Sigma\, m(r + a)^2$

$$I = \Sigma\, m(r^2 + a^2 + 2ar)$$

$$= \Sigma\, mr^2 + a^2\, \Sigma m + 2a\, \Sigma mr$$

ie. $$I = I_G + Ma^2 + 2a\, \Sigma mr$$

Σmr represents the algebraic sum of the moments of all the mass particles of the body about an axis through the centre of gravity of the body. Since the body always balances about an axis through its centre of gravity. Σmr should be zero.

$$\text{Therefore, } I = I_G + Ma^2$$

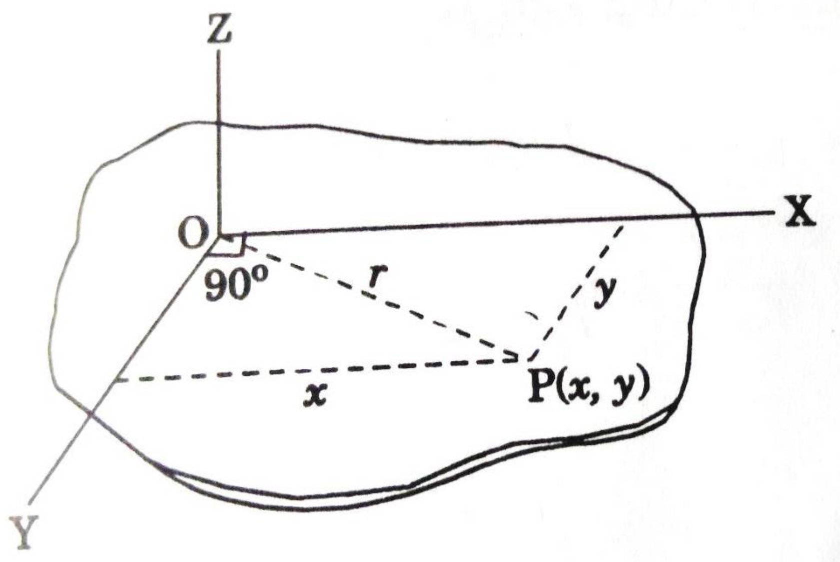
2. Theorem (Principle) of Perpendicular axes

Statement

It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the plane lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through it.

Explanation

Let OX and OY be two mutually perpendicular axes in the plane of the lamina, intersecting each other at the point O. OZ is the axes perpendicular to both OX and OY.



Let I_x and I_y be the moments of inertia of the lamina about the axis OX and OY respectively.

If I_z is the moment of inertia about the axis OZ, passing through O and perpendicular to the plane of the lamina, then by the theorem of perpendicular axes,

$$I_z = I_x + I_y$$

Proof

Consider a particle P of the lamina of mass m at a distance r from O. let x and y be the distances of the particle from OY and OX respectively.

M.I. of this particle about OX = my^2

M.I. of the entire lamina about OX, $I_x = my^2$

Similarly M.I. of the lamina about OY, $I_y = mx^2$

M.I. of the lamina about OZ axis through O and perpendicular to the plane lamina,

$$I_G = \Sigma mr^2 \quad \text{---------- (1)}$$

But $$r^2 = x^2 + y^2 \quad \text{---------- (2)}$$

Substituting eqn (2) in eqn (1), we have

$$I_z = \Sigma m (x^2 + y^2)$$

$$I_z = \Sigma m x^2 + \Sigma m y^2$$

$$I_z = I_y + I_x$$

ie. $$I_z = I_x + I_y$$

Calculation of Moment of Inertia of a body

The moment of inertia of a continuous homogeneous body (Rigid body) with definite geometrical shape can be calculated as follows.

- Find the moment of inertia of an infinitesimal element of the body about the given axis. ie., Multiply the mass dm of the element by x^2, the square of the distance from the given axis.
- Then, integrate the expression between the limits to get moment of inertia of whole of the body.

$$I = \int dm.\, x^2$$

Where the integral is taken over the whole body.

In fact sometimes the theorems parallel and perpendicular axes are also used to calculate the moment of inertia.