1.3 Pigeonhole Principle

Introduction

The **Pigeonhole Principle** is a simple, yet powerful concept in combinatorics and mathematics. It provides insight into problems that might initially seem counterintuitive or difficult to solve. It's based on an extremely basic idea, but its applications can be vast, ranging from number theory to computer science and beyond.

The Pigeonhole Principle

Statement:

- If you have more **objects** than **containers** and you try to distribute the objects into the containers, then at least one container must hold more than one object.

Formal Version:

- If nnn objects are placed into mmm containers, and n>mn > mn>m, then at least one container must hold at least ⌈nm⌉\lceil \frac{n}{m} \rceil⌈mn⌉ objects.

Visualizing the Principle

Imagine you have **pigeons** and **pigeonholes**. If you have more pigeons than pigeonholes and try to place the pigeons in the holes, at least one pigeonhole will contain more than one pigeon.

Example:

- If there are 11 pigeons and only 10 pigeonholes, then at least one pigeonhole must contain at least 2 pigeons.

Generalization and Extended Versions

- **Basic Principle:** If n>mn > mn>m, then at least one pigeonhole must contain more than one pigeon.
- **Generalized Version**: If nnn objects are placed into mmm containers, then at least one container contains at least ⌈nm⌉\left\lceil \frac{n}{m} \right\rceil⌈mn⌉ objects.

Applications and Examples

1. **Example 1: Distribution of Balls into Boxes**

   Problem: You have 15 balls and 12 boxes. What is the minimum number of balls that must be in one box?

   Solution: By the pigeonhole principle, if you place the 15 balls into 12 boxes, at least one box must contain ⌈1512⌉=2\lceil \frac{15}{12} \rceil = 2⌈1215⌉=2 balls. Hence, the answer is 2 balls.

2. **Example 2: Birthdays in a Group**

   Problem: In a group of 23 people, what is the probability that at least two people share the same birthday?

   Solution: We have 365 possible birthdays (ignoring leap years), and 23 people. Using the pigeonhole principle, since 23

The **Pigeonhole Principle** is a simple yet powerful concept in combinatorics. It states that if you place more items into fewer containers than the number of items, at least one container must hold more than one item. In more formal terms:

- **Pigeonhole Principle**: If nnn items are put into mmm containers, and n>mn > mn>m, then at least one container must contain more than one item.

In set theory and combinatorics, this principle is used to prove the existence of certain configurations or distributions. Let's look at some problems involving the Pigeonhole Principle in set theory.

Problem 1: Basic Application

Problem: In a set of 10 people, there are only 4 distinct hair colors. Prove that at least two people must have the same hair color.

Solution:

- Let the 10 people be the "items" to be placed in "containers," and the 4 distinct hair colors be the containers.
- By the Pigeonhole Principle, if you place 10 people (items) into 4 hair colors (containers), and 10 > 4, then at least one hair color (container) must be shared by more than one person.
- Therefore, at least two people must have the same hair color.

Problem 2: Distributing Elements in Subsets

Problem: Given a set S={1,2,3,4,5,6,7,8,9,10}S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}S={1,2,3,4,5,6,7,8,9,10}, how many subsets of size 3 can be chosen such that every subset has at least two elements that are divisible by 3?

Solution:

1. First, let's identify the elements of SSS that are divisible by 3. These elements are {3,6,9}\{3, 6, 9\}{3,6,9}.
2. The remaining elements of SSS are {1,2,4,5,7,8,10}\{1, 2, 4, 5, 7, 8, 10\}{1,2,4,5,7,8,10}, which are not divisible by 3.
3. To ensure that each subset of size 3 contains at least two elements divisible by 3, we must select at least two elements from {3,6,9}\{3, 6, 9\}{3,6,9}, and the third element can be chosen from the remaining set {1,2,4,5,7,8,10}\{1, 2, 4, 5, 7, 8, 10\}{1,2,4,5,7,8,10}.
4. The number of ways to choose two elements from {3,6,9}\{3, 6, 9\}{3,6,9} is (32)=3\binom{3}{2} = 3(23)=3.
5. After choosing two elements from {3,6,9}\{3, 6, 9\}{3,6,9}, the third element must come from the set of 7 elements {1,2,4,5,7,8,10}\{1, 2, 4, 5, 7, 8, 10\}{1,2,4,5,7,8,10}, and there are 7 possible choices for the third element.
6. Therefore, the total number of subsets is 3×7=213 \times 7 = 213×7=21.

Problem 3: Generalized Pigeonhole Principle

Problem: Prove that if nnn items are placed into mmm containers, and n>kmn > kmn>km, then at least one container contains more than kkk items.

Solution:

- This is a generalization of the basic Pigeonhole Principle.
- The idea is that if nnn items are distributed among mmm containers, and there are more than kmkmkm items (i.e., n>kmn > kmn>km), then there must be at least one container with more than kkk items.
- If each container could hold at most kkk items, the maximum number of items would be kmkmkm. Since n>kmn > kmn>km, we can conclude that one of the containers must hold more than kkk items.

Problem 4: Set Partition Problem

Problem: Let S={1,2,3,4,5,6}S = \{1, 2, 3, 4, 5, 6\}S={1,2,3,4,5,6}. Show that there is a way to partition SSS into two subsets such that the sum of the elements in each subset is equal.

Solution:
This is a classic example of applying the Pigeonhole Principle in a set partition context. We want to find a partition of SSS into two subsets such that the sums of the elements in each subset are equal.

- First, compute the sum of all elements in SSS:
  Sum of S=1+2+3+4+5+6=21\text{Sum of } S = 1 + 2 + 3 + 4 + 5 + 6 = 21Sum of S=1+2+3+4+5+6=21
- For two subsets to have equal sums, each subset must have a sum of 212=10.5\frac{21}{2} = 10.5221=10.5, which is not possible because the sum must be an integer. Therefore, it's impossible to partition SSS into two subsets with equal sums.

This shows that the Pigeonhole Principle can be used to determine the impossibility of certain partitions.

Problem 5: Application to Functions

Problem: Consider the set A={1,2,3,4}A = \{1, 2, 3, 4\}A={1,2,3,4} and the set B={a,b,c}B = \{a, b, c\}B={a,b,c}. Let f:A→Bf: A \to Bf:A→B be a function. Prove that if fff is a surjection, then at least one element in AAA must map to the same element in BBB.

Solution:

- Since fff is a surjection, every element in BBB must be the image of some element in AAA.
- The set AAA has 4 elements, and the set BBB has 3 elements. By the Pigeonhole Principle, if you map 4 elements to 3 containers (elements of BBB), at least one container (element of BBB) must contain more than one element from AAA.
- Therefore, at least one element in AAA must map to the same element in BBB.

Conclusion

The Pigeonhole Principle is a useful tool in set theory and combinatorics. By applying it to various situations, you can prove the existence of certain configurations (e.g., equal sums, overlapping elements) or demonstrate the impossibility of certain outcomes.

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