1.4 OPERATIONS ON SETS

Operations on Sets (Set Theory)

Set theory is the foundation of modern mathematics. It deals with the study of sets, which are collections of well-defined objects. The objects in a set are called elements or members of the set. The operations on sets are the tools we use to manipulate or combine sets to create new sets or extract information from them.

Basic Set Operations

Here are the most common operations on sets:

Union of Sets (A∪BA \cup BA∪B)

The union of two sets AAA and BBB is the set of all elements that belong to either AAA, or BBB, or both.

Definition:

A∪B={x∣x∈A or x∈B}A \cup B = \{ x \mid x \in A \text{ or } x \in B \}A∪B={x∣x∈A or x∈B}

- **Example**:
  Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. Then: A∪B={1,2,3,4,5}A \cup B = \{ 1, 2, 3, 4, 5 \}A∪B={1,2,3,4,5} Note that the element 3 appears only once, even though it is in both sets.

Intersection of Sets (A∩BA \cap BA∩B)

The intersection of two sets AAA and BBB is the set of all elements that are common to both AAA and BBB.

Definition:

A∩B={x∣x∈A and x∈B}A \cap B = \{ x \mid x \in A \text{ and } x \in B \}A∩B={x∣x∈A and x∈B}

- **Example**:
  Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. Then: A∩B={3}A \cap B = \{ 3 \}A∩B={3}

Difference of Sets (A−BA - BA−B or A\BA \setminus BA\B)

The difference of two sets AAA and BBB is the set of all elements that belong to AAA but not to BBB.

Definition:

A−B={x∣x∈A and x∉B}A - B = \{ x \mid x \in A \text{ and } x \notin B \}A−B={x∣x∈A and x∈/B}

- **Example**:
  Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. Then: A−B={1,2}A - B = \{ 1, 2 \}A−B={1,2}

Complement of a Set (AcA^cAc)

The complement of a set AAA refers to the set of all elements in the universal set UUU that are not in AAA.

Definition:

Ac={x∣x∈U and x∉A}A^c = \{ x \mid x \in U \text{ and } x \notin A \}Ac={x∣x∈U and x∈/A}

where UUU is the universal set containing all possible elements under consideration.

- **Example**:
  Let the universal set U={1,2,3,4,5}U = \{ 1, 2, 3, 4, 5 \}U={1,2,3,4,5} and A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3}. Then: Ac={4,5}A^c = \{ 4, 5 \}Ac={4,5}

Symmetric Difference of Sets (A△BA \triangle BA△B)

The symmetric difference between two sets AAA and BBB is the set of elements that belong to either AAA or BBB, but not to both.

Definition:

A▵B=(A−B)∪(B−A)A \triangle B = (A - B) \cup (B - A)A▵B=(A−B)∪(B−A)

or equivalently,

A▵B=(A∪B)−(A∩B)A \triangle B = (A \cup B) - (A \cap B)A▵B=(A∪B)−(A∩B)

- **Example**:
  Let A={1,2,3}A = \{ 1, 2, 3 \}A={1,2,3} and B={3,4,5}B = \{ 3, 4, 5 \}B={3,4,5}. Then: A▵B={1,2,4,5}A \triangle B = \{ 1, 2, 4, 5 \}A▵B={1,2,4,5}

Properties of Set Operations

Commutative Property

- **Union**: A∪B=B∪AA \cup B = B \cup AA∪B=B∪A
- **Intersection**: A∩B=B∩AA \cap B = B \cap AA∩B=B∩A

Associative Property

- **Union**: (A∪B)∪C=A∪(B∪C)(A \cup B) \cup C = A \cup (B \cup C)(A∪B)∪C=A∪(B∪C)
- **Intersection**: (A∩B)∩C=A∩(B∩C)(A \cap B) \cap C = A \cap (B \cap C)(A∩B)∩C=A∩(B∩C)

Distributive Property

- **Intersection over Union**: A∩(B∪C)=(A∩B)∪(A∩C)A \cap (B \cup C) = (A \cap B) \cup (A \cap C)A∩(B∪C)=(A∩B)∪(A∩C)
- **Union over Intersection**: A∪(B∩C)=(A∪B)∩(A∪C)A \cup (B \cap C) = (A \cup B) \cap (A \cup C)A∪(B∩C)=(A∪B)∩(A∪C)

Idempotent Property

- **Union**: A∪A=AA \cup A = AA∪A=A
- **Intersection**: A∩A=AA \cap A = AA∩A=A

Complementation

- (Ac)c=A(A^c)^c = A(Ac)c=A (Double complement)
- A∪Ac=UA \cup A^c = UA∪Ac=U (The union of a set and its complement is the universal set)
- A∩Ac=∅A \cap A^c = \emptysetA∩Ac=∅ (The intersection of a set and its complement is the empty set)

De Morgan's Laws

- (A∪B)c=Ac∩Bc(A \cup B)^c = A^c \cap B^c(A∪B)c=Ac∩Bc
- (A∩B)c=Ac∪Bc(A \cap B)^c = A^c \cup B^c(A∩B)c=Ac∪Bc

Venn Diagrams

Venn diagrams are a helpful tool for visualizing set operations. They represent sets as circles and show the relationships between sets using overlapping or non-overlapping regions.

- **Union**: The area covered by both circles.
- **Intersection**: The overlapping area between the circles.
- **Difference**: The part of one circle that does not overlap with the other.
- **Complement**: The area outside a circle (in the universal set).

Advanced Set Operations

Cartesian Product (A×BA \times BA×B)

The Cartesian product of two sets AAA and BBB is the set of all ordered pairs where the first element comes from AAA and the second comes from BBB.

Definition:

A×B={(a,b)∣a∈A,b∈B}A \times B = \{ (a, b) \mid a \in A, b \in B \}A×B={(a,b)∣a∈A,b∈B}

- **Example**:
  Let A={1,2}A = \{ 1, 2 \}A={1,2} and B={a,b}B = \{ a, b \}B={a,b}. Then: A×B={(1,a),(1,b),(2,a),(2,b)}A \times B = \{ (1, a), (1, b), (2, a), (2, b) \}A×B={(1,a),(1,b),(2,a),(2,b)}

Power Set (P(A)P(A)P(A))

The power set of a set AAA is the set of all subsets of AAA, including the empty set and AAA itself.

Definition:

P(A)={B∣B⊆A}P(A) = \{ B \mid B \subseteq A \}P(A)={B∣B⊆A}

- **Example**:
  Let A={1,2}A = \{ 1, 2 \}A={1,2}. Then: P(A)={∅,{1},{2},{1,2}}P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}P(A)={∅,{1},{2},{1,2}}

Applications of Set Theory

- **Database Theory**: Sets are used to model relations and queries.
- **Logic**: Set operations are used to express logical connectives (e.g., AND, OR, NOT).
- **Computer Science**: Sets are crucial for data structures, searching algorithms, and problem-solving.
- **Probability Theory**: Events in probability are represented as sets, with operations defining events like unions, intersections, and complements.