4.3 Techniques of integration

Integration by parts

If the integrand is either a product or quotient of polynomial and a transcendental function such as trigonometric, exponential or logarithmic function then have to develop different methods to evaluate them.

$$\int f(x)g'(x)dx = f(x)\, g(x) - \int g(x)f'(x)dx$$

The above formula is called the integration by parts.

Let if we take $u = f(x)$ and $v = g(x)$

Then the above formula becomes $\int udv = uv - \int vdu$

To choose u, we should follow the following order

I – Inverse function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

Note:

The generalized integration by parts formula is known as Bernoulli's formula

Bernoulli's formula states that

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Where $v_1, v_2, v_3, \ldots$are functions obtained by integrating v successively with respect to x and u', u'', ... are functions obtained by differentiating u successively with respect to x.

Example:

Evaluate $\int xe^{-x}dx$

Solution:

Let $u = x$ $\qquad dv = e^{-x}dx$

$du = dx$ $\qquad v = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int xe^{-x}dx = x(-e^{-x}) - \int -e^{-x}\, dx$$

$$= -xe^{-x} + \int e^{-x}\, dx$$

$$= -xe^{-x} + (-e^{-x}) + C = -(xe^{-x} + e^{-x} + C)$$

$$= -e^{-x}(x+1) + C$$

Example :

Evaluate $\int x^4 logxdx$

Solution:

Let $u = logx$ $\qquad dv = x^4dx$

$$du = \frac{1}{x}dxv = \int x^4dx = \frac{x^5}{5}$$

$$\int udv = uv - \int vdu$$

$$\int x^4logxdx = (logx)\left(\frac{x^5}{5}\right) - \int \frac{x^5}{5}\frac{1}{x}dx$$

$$= \frac{x^5}{5}logx - \frac{1}{5}\int x^4dx$$

$$= \frac{x^5}{5}logx - \frac{1}{5}\frac{x^5}{5} + C$$

$$= \frac{x^5}{5}logx - \frac{x^5}{25} + C$$

Example :

Evaluate $\int(\log x)^2$ dx

Solution:

Let $u = (\log x)^2$ $\qquad dv = dx$

$du = 2\ logx\left(\frac{1}{x}\right)dx$ $\qquad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int(\log x)^2dx = (\log x)^2x - \int\left(x\ 2\ logx\left(\frac{1}{x}\right)dx\right)$$

$$= x(\log x)^2 - 2\int logxdx \ldots (1)$$

Take $\int logxdx$

Let $u = logx$ $\qquad dv = dx$

$du = \frac{1}{x}dx$ $\qquad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int logxdx = (logx)(x) - \int x\frac{1}{x}dx$$

$$= xlogx - \int dx = x\,logx - x$$

$$(1) \Rightarrow \int (\log x)^2 dx = x(\log x)^2 - 2\,[xlogx - x] + C$$

Example :

Evaluate $\int xsec^2 2xdx$

Solution:

Let $u = x \qquad dv = sec^2 2xdx$

$$du = dxv = \int sec^2 2xdx = \frac{\tan 2x}{2}$$

$$\int udv = uv - \int vdu$$

$$\int xsec^2 2xdx = (x)\left(\frac{\tan 2x}{2}\right) - \int \frac{\tan 2x}{2} dx$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{2}\int \tan 2xdx$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{2}\left[\frac{\log(\sec 2x)}{2}\right] + C$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{4}\log(\sec 2x) + C$$

Example :

Evaluate $\int xsin^2 xdx$

Solution:

Let $u = x \qquad dv = sin^2 xdx$

$du = dx \qquad v = \frac{1}{2}\int(1 - cos2x)dx = \frac{1}{2}\left(x - \frac{sin2x}{2}\right)$

$$\int udv = uv - \int vdu$$

$$\int xsin^2 xdx = \frac{x}{2}\left(x - \frac{sin2x}{2}\right) - \frac{1}{2}\int\left(x - \frac{sin2x}{2}\right)dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\int\left(x - \frac{sin2x}{2}\right)dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{cos2x}{4}\right) + C$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{x^2}{4} - \frac{cos2x}{8} + C$$

$$= \frac{x^2}{4} - \frac{xsin2x}{4} - \frac{cos2x}{8} + C$$

Example :

Evaluate $\int \frac{x}{1+cosx} dx$

Solution:

$$\int \frac{x}{1+cosx}dx = \int \frac{x}{2cos^2\frac{x}{2}}dx \qquad \left[\because 1+cosx = 2cos^2\frac{x}{2}\right]$$

$$= \frac{1}{2}\int xsec^2\frac{x}{2}dx \cdots (1)$$

Let $u = x$ $\qquad dv = sec^2\frac{x}{2}dx$

$du = dx$ $\qquad v = \int sec^2\frac{x}{2}dx = \frac{\tan\left(\frac{x}{2}\right)}{\frac{1}{2}} = 2\tan\frac{x}{2}$

$$\int udv = uv - \int vdu$$

$$(1) \Rightarrow \int \frac{x}{1+cosx}dx = \frac{1}{2}\left[x\left(2\tan\frac{x}{2}\right) - \int 2\tan\frac{x}{2}dx\right]$$

$$= x\tan\frac{x}{2} - \frac{log\left[\sec\left(\frac{x}{2}\right)\right]}{\frac{1}{2}} + C$$

$$= x\tan\frac{x}{2} - 2\,log\left[\sec\left(\frac{x}{2}\right)\right] + C$$

Example :

Evaluate $\int \frac{x}{1+sinx}dx$

Solution:

$$\int \frac{x}{1+sinx}dx = \int \frac{x\,(1-sinx)}{(1+sinx)(1-sinx)}dx$$

$$= \int \frac{x\,(1-sinx)}{1-sin^2x}dx = \int \frac{x\,(1-sinx)}{cos^2x}dx$$

$$= \int (xsec^2x - xsecxtanx)\,dx$$

$$= \int xsec^2xdx - \int xsecxtanxdx \;\ldots (1)$$

Take $\int xsec^2xdx$

Let $u = x$ $\qquad dv = sec^2xdx$

$du = dx$ $\qquad v = \int sec^2xdx = tanx$

$$\int udv = uv - \int vdu$$

$$\int xsec^2xdx = (x)(tanx) - \int tanxdx$$

$$= xtanx - \log(secx) \ldots (2)$$

Take $\int xsecxtanxdx$

Let $u = x$ $\qquad dv = secxtanxdx$

$du = dx$ $\qquad v = \int secxtanxdx = secx$

$$\int udv = uv - \int vdu \quad = \int xsecxtanxdx = (x)(secx) - \int secxdx$$

$$= xsecx - \log(secx + \tan x) \quad \dots (3)$$

$$(1) \Rightarrow \int \frac{x}{1+sinx} dx = xtanx - \log(secx) - xsecx + \log(secx + tanx) + C$$

$$[\because by(2)and(3)\,]$$

Example :

Evaluate $\int (x^2 e^{2x})\, dx$

Solution:

Let $u = x^2$, $u' = 2x,$ $u'' = 2,$

$$dv = e^{2x}dx,\ \ v = \frac{e^{2x}}{2}, \quad v_1 = \frac{e^{2x}}{4}, \quad v_2 = \frac{e^{2x}}{8}$$

$$\int udv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int (x^2 e^{2x})\, dx = (x^2)\frac{e^{2x}}{2} - (2x)\frac{e^{2x}}{4} + (2)\frac{e^{2x}}{8} + \text{C}$$

$$= (x^2)\frac{e^{2x}}{2} - (x)\frac{e^{2x}}{2} + \frac{e^{2x}}{4} + \text{C}$$

Example :

Evaluate $\int e^x cosxdx$

Solution:

Let $u = e^x$ $\quad dv = cosxdx$

$du = e^x dx$ $\quad v = \int cosxdx = sinx$

$$\int udv = uv - \int vdu$$

$$\text{I} = \int e^x cosxdx = e^x sinx - \int sinxe^x dx \quad \dots (1)$$

Take $\int e^x sinxdx$

Let $u = e^x$ $\quad dv = \sin xdx$

$du = e^x dx$ $\quad v = \int sinxdx = -cosx$

$$\int udv = uv - \int vdu$$

$$\int e^x sinxdx = (e^x)(-cosx) - \int (-cosx)(e^x)\, dx$$

$$= -e^x cosx + \int e^x cosxdx = -e^x cosx + I$$

$$(1) \Rightarrow I = e^x sinx - [-e^x cosx + I\,] + C$$

$$I = e^x sinx + e^x cosx - I + C$$

$$2I = e^x sinx + e^x cosx + C$$

$$I = \frac{1}{2}[e^x sinx + e^x cosx] + C$$

$$\therefore \int e^x cosxdx = \frac{e^x}{2}[sinx + cosx] + C$$

Example :

Evaluate $\int e^{2x} sinxdx$

Solution:

$$I = \int e^{2x} sinxdx \quad ...(1)$$

Let $u = sinx$ $\quad\quad dv = e^{2x}dx$

$du = cosx\,dx$ $\quad\quad v = \frac{e^{2x}}{2}$

$$\int udv = uv - \int vdu$$

$$I = sinx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} cosx\,dx$$

$$= \frac{e^{2x}}{2} sinx - \frac{1}{2} I_1 \ ...(2)$$

Take $I_1 = \int e^{2x} \cos x\, dx$

Let $u = cosx$ $\quad\quad dv = e^{2x}dx$

$du = -sinxdx$ $\quad\quad v = \frac{e^{2x}}{2}$

$$I_1 = cosx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(-sinx)dx$$

$$= \frac{e^{2x}}{2} cosx + \frac{1}{2}\int e^{2x} sinxdx$$

$$= \frac{e^{2x}}{2} cosx + \frac{1}{2} I$$

$$(2) \Rightarrow I = \frac{e^{2x}}{2} sinx - \frac{1}{2}\left[\frac{e^{2x}}{2} cosx + \frac{1}{2} I\right]$$

$$I = \frac{e^{2x}}{2} sinx - \frac{e^{2x}}{4} cosx - \frac{1}{4} I$$

$$I + \frac{1}{4} I = \frac{e^{2x}}{2} sinx - \frac{e^{2x}}{4} cosx$$

$$\frac{5}{4} I = \frac{e^{2x}}{4}(2\,sinx - cosx)$$

$$\therefore \quad I = \frac{e^{2x}}{5}(2\,sinx - cosx) + C$$

Example :

Evaluate $\int \boldsymbol{tan^{-1}xdx.\ Alsofind \int_0^1 tan^{-1}xdx}$

Solution:

Let $u = tan^{-1}x$ $\qquad dv = dx$

$du = \frac{1}{1+x^2}dx v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int tan^{-1}xdx = xtan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$$

$$= xtan^{-1}x - \int\left(\frac{x}{1+x^2}\right)dx \ldots (1)$$

Take $\int\left(\frac{x}{1+x^2}\right)dx$

Put $t = 1 + x^2$, $dt = 2xdx$

$$\int\left(\frac{x}{1+x^2}\right)dx = \int \frac{1}{t}\frac{1}{2}dt = \frac{1}{2}\int\frac{1}{t}dt = \frac{1}{2}logt = \frac{1}{2}\log(1+x^2)$$

$$(1) \Rightarrow \int tan^{-1}xdx = xtan^{-1}x - \frac{1}{2}\log(1+x^2) + C \ \ldots (2)$$

To find $\int_0^1 tan^{-1}x$

$$(2) \Rightarrow \int_0^1 tan^{-1}x = [xtan^{-1}x]_0^1 - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= tan^{-1}1 - 0 - \left[\frac{1}{2}log2 - \frac{1}{2}log1\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}log2 \qquad [\because \log 1 = 0]$$

Reduction Formula

(I) Find the reduction formula for $\int sin^n x\, dx\,; n \geq 2$ is an integer

Solution:

Consider $I_n = \int sin^n x\, dx = \int sin^{n-1}x\, sinx\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = sin^{n-1}x$ $\qquad dv = sinx\, dx;$

$du = (n-1)sin^{n-2}x\, cosx\, dx \qquad v = \int sinx\, dx = -cosx$

$$I_n = -cosx\, sin^{n-1}x - \int(-cosx)(n-1)sin^{n-2}\, x\, cosx\, dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int cos^2 x sin^{n-2}x \; dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int (1 - sin^2 x)\, sin^{n-2}x \; dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int sin^{n-2}x \; dx - (n-1) \int sin^n x \, dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int sin^{n-2}x \; dx - (n-1) I_n$$

$$I_n + (n-1)I_n = -cosx\, sin^{n-1}x + (n-1) \int sin^{n-2}x \; dx$$

$$nI_n = -cosx\, sin^{n-1}x + (n-1) \int sin^{n-2}x \; dx$$

$$I_n = -\frac{cosx\, sin^{n-1}x}{n} + \frac{(n-1)}{n} \int sin^{n-2}x \; dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int sinx\, dx = -cosx + C$ [Put n = 1 in (1)]

(II) Find the reduction formula for $\int cos^n x\, dx; n \geq 2$ is an integer

Solution:

Consider $I_n = \int cos^n x\, dx = \int cos^{n-1}x\, cosx\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = cos^{n-1}x$ $\qquad dv = cosx\, dx$

$du = (n-1)cos^{n-2}x\,(-sinx)dx$ $\qquad v = \int cosx\, dx = sinx$

$$I_n = sinx\, cos^{n-1}x - \int (sinx)[-(n-1)cos^{n-2} x\, sinx]\, dx$$

$$= sinx\, cos^{n-1}x + (n-1) \int sin^2 x cos^{n-2}x \; dx$$

$$= sinx\, cos^{n-1}x + (n-1) \int (1 - cos^2 x)\, cos^{n-2}x \; dx$$

$$= sinx\, cos^{n-1}x + (n-1) \int cos^{n-2}x \; dx - (n-1) \int cos^n x\, dx$$

$$= sinx\, cos^{n-1}x + (n-1) \int cos^{n-2}x \; dx - (n-1)I_n$$

$$I_n + (n-1)I_n = sinx\, cos^{n-1}x + (n-1) \int cos^{n-2}x \; dx$$

$$nI_n = sinx\, cos^{n-1}x + (n-1) \int cos^{n-2}x \; dx$$

$$I_n = \frac{sinx\, cos^{n-1}x}{n} + \frac{(n-1)}{n} \int cos^{n-2}x \; dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int cosx\, dx = sinx + C$ [Put n = 1 in (1)]

(III)Find the reduction formula for $\int_0^{\pi/2} sin^n x\, dx$

Solution:

Consider $I_n = \int_0^{\pi/2} sin^n x\, dx$

We know that $\int sin^n x\, dx = -\frac{cosx\, sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\ dx$

$$\int_0^{\pi/2} sin^n x\, dx = \left[-\frac{cosx\, sin^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2}\, x\, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2}\, x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} sin^{n-4}\, x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} sin^{n-6}\, x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots I$$

If n is even then,

$$I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

If n is odd then,

$$I = \int_0^{\pi/2} sinx\, dx = (-cosx)_0^{\pi/2} = -cos\,\frac{\pi}{2} + cos0 = 0 + 1 = 1$$

Thus,

$$\int_0^{\pi/2} sin^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if\ \ n\ is\ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1, & if\ n\ is\ odd \end{cases}$$

(IV)Find the reduction formula for $\int_0^{\pi/2} cos^n x\, dx$

Solution:

Consider $I_n = \int_0^{\pi/2} cos^n x\, dx$

We know that $\int cos^n x\, dx = \frac{sinx\, cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\ dx$

$$\int_0^{\pi/2} cos^n x\, dx = \left[\frac{sinx\, cos^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2}\, x\, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2}\, x\, d$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} cos^{n-4}\, x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} cos^{n-6}\, x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots I$$

If n is even then,

$$I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

If n is odd then,

$$I = \int_0^{\pi/2} cosx\, dx = (sinx)_0^{\pi/2} = sin\,\frac{\pi}{2} - sin0 = 1 - 0 = 1$$

Thus,

$$\int_0^{\pi/2} cos^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if \quad n\, is\ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1, & if\ n\ is\ odd \end{cases}$$

(V) Find the reduction formula for $\int sec^n\, xdx,\ n \geq 2$ is an integer.

Solution:

Consider $I_n = \int sec^n\, xdx = \int sec^{n-2}\, xsec^2xdx$ ... (1)

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = sec^{n-2}x$ $\qquad dv = sec^2x\, dx$

$du = (n-2)cos^{n-3}x\,(secxtanx)dx \qquad v = \int sec^2x\, dx = tanx$

$$I_n = sec^{n-2}x\, tanx - \int (tanx)[(n-2)sec^{n-3}\, x\, secx\, tanx]\, dx$$

$$= sec^{n-2}x\, tanx - (n-2)\int tan^2\, x\, sec^{n-2}x\ dx$$

$$= sec^{n-2}x\, tanx - (n-2)\int (sec^2x - 1)\, sec^{n-2}x\ dx$$

$$= sec^{n-2}x\, tanx - (n-2)\int sec^n x\ dx + (n-2)\int sec^{n-2}x\, dx$$

$$= sec^{n-2}x\, tanx - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = sec^{n-2}x\, tanx + (n-2)I_{n-2}$$

$$(n-1)I_n = sec^{n-2}x\, tanx + (n-2)I_{n-2}$$

$$I_n = \frac{1}{n-1}sec^{n-2}x\, tanx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is I_0 or I_1

$n\ even \quad : \quad I_0 \ = \ \int dx \ = x + C$ [Put n = 0 in (1)]

$n\ odd \quad : \quad I_1 \ = \ \int secx\ dx \ = \log(secx + tanx) + C$ [Put n = 1 in (1)]

(VI) Find the reduction formula for $\int cosec^n\, xdx,\ n \geq 2$ is an integer.

Solution:

Consider $I_n = \int cosec^n\, xdx = \int cosec^{n-2}\, x\, cosec^2 xdx \ \ldots (1)$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = \ cosec^{n-2}x$ $\qquad dv = cosec^2 x\, dx$

$$du = (n-2)sec^{n-3}x\,(-\,cosecx\, cotx)dx$$

$$v = \ \int cosec^2 x\, dx = \ -cotx$$

$$I_n = \ cosec^{n-2}x\,(-cotx) - \ \int(-cotx)[(n-2)cosec^{n-3}\, x\,(-cosecx\, cotx)]\, dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\ \int cot^2\, x\, cosec^{n-2}x\ \ dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\ \int(cosec^2x - 1)\ cosec^{n-2}x\ \ dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\int\ cosec^n x\ \ dx + (n-2)\int cosec^{n-2}x\, dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = \ -cosec^{n-2}x\, cotx + (n-2)I_{n-2}$$

$$(n-1)I_n \ = \ -cosec^{n-2}x\, cotx + (n-2)I_{n-2}$$

$$I_n \ = \ -\frac{1}{n-1}cosec^{n-2}x\, cotx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is I_0 or I_1

$n\ even \quad : \quad I_0 \ = \ \int dx \ = x + C$ [Put n = 0 in (1)]

$n\ odd \quad : \quad I_1 \ = \ \int cosecx\ dx \ = \log(cosecx - cotx) + C$ [Put n = 1 in (1)]

(VII) Find the reduction formula for $\int cot^n\, xdx,\ n \neq 1$

Solution:

Consider $I_n = \int cot^n\, xdx = \int cot^{n-2}x\, cot^2 x\, dx \ \ldots (1)$

$$= \int cot^{n-2}x\,(cosec^2x - 1)\, dx$$

$$= -\int cot^{n-2}x\,(cosec^2x)dx - \int cot^{n-2}x\, dx$$

$$= -\int\ cot^{n-2}x\ \ d(cotx) - I_{n-2}$$

$$= -\frac{1}{n-1} cot^{n-1}x - I_{n-2}$$

The ultimate integral is I_0 or I_1

n $even$: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd : $I_1 = \int cotx\, dx = \log(sinx) + C$ [Put n = 1 in (1)]

(VIII)Find the reduction formula for $\int tan^n xdx,\ n \neq 1$

Solution:

Consider $I_n = \int tan^n xdx$ ... (1)

$$= \int tan^{n-2}x\, tan^2x\, dx$$

$$= \int tan^{n-2}x\, (sec^2x - 1)\, dx$$

$$= \int tan^{n-2}x\, sec^2xdx - \int tan^{n-2}x\, dx$$

$$= \int tan^{n-2}x\ d(tanx) - I_{n-2}$$

$$= \frac{1}{n-1} tan^{n-1}x - I_{n-2}$$

The ultimate integral is I_0 or I_1

n $even$: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd : $I_1 = \int tanx\, dx = \log(secx) + C$ [Put n = 1 in (1)]

Example:

i) Evaluate $\int sin^7x\, dx$

Solution:

Given $\int sin^7x\, dx$

We know that $I_n = -\frac{cosx\, sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\ dx \cdots$ (1)

Put n = 7 in equation (1)

$$\int sin^7x\, dx = -\frac{cosx\, sin^{7-1}x}{7} + \frac{7-1}{7}\int sin^{7-2}x\ dx$$

$$\int sin^7x\, dx = -\frac{cosx\, sin^6x}{7} + \frac{6}{7}\int sin^5x\ dx \cdots (2)$$

Put n = 5 in equation (1)

$$\int sin^5x\, dx = -\frac{cosx\, sin^4x}{5} + \frac{4}{5}\int sin^3x\ dx \cdots (3)$$

Put n = 3 in equation (1)

$$\int sin^5x\, dx = -\frac{cosx\, sin^2x}{3} + \frac{2}{3}\int sinx\ dx$$

$$= -\frac{cosx\ sin^2x}{3} + \frac{2}{3}(-cosx)$$

$$\therefore (3) \quad gives \int sin^5x\ dx = -\frac{cosx\ sin^4x}{5} + \frac{4}{5}\left[\frac{-sin^2x\ cosx}{3} - \frac{2}{3}cosx\right]$$

$$= -\frac{cosx\ sin^4x}{5} - \frac{4}{15}sin^2x\ cosx - \frac{8}{15}\ cosx$$

and (2) gives

$$\int sin^7x\ dx = -\frac{cosx\ sin^6x}{7} + \frac{6}{7}\left[\frac{-sin^4x\ cosx}{5} - \frac{4}{15}sin^2\ x\ cosx - \frac{8}{15}\ cosx\right]$$

$$= -\frac{1}{7}cosx\ sin^6x - \frac{6}{35}sin^4x\ cosx - \frac{8}{35} - sin^2x\ cosx - \frac{16}{35}cosx$$

(ii) Evaluate $\int cos^4x\ dx$

Solution:

Given $\int cos^4x\ dx$

We know that $I_n = \frac{sin\,x\ cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\ \ dx \quad \cdots (1)$

Put n = 4 in equation (1)

$$\int cos^4x\ dx = \frac{sin\,x\ cos^3x}{4} + \frac{3}{4}\int cos^2x\ \ dx$$

$$= \frac{sin\,x\ cos^3x}{4} + \frac{3}{4}\int\left(\frac{1+cos\,2x}{2}\right)dx$$

$$= \frac{sin\,x\ cos^3x}{4} + \frac{3}{8}\left(x + \frac{\sin 2x}{2}\right)$$

(iii)Evaluate $\int_0^{\pi/2} sin^7\ x\ dx$

Solution:

Given $\int_0^{\pi/2} sin^7\ x\ dx$

We know that $\int_0^{\pi/2} sin^n\ x\ dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}.1,\ when\ n\ is\ odd \quad \cdots (1)$

Put n = 7 in equation (1)

$$\int_0^{\pi/2} sin^7\ x\ dx = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)(1)$$

$$= \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1)$$

(iv)Evaluate $\int_0^{\pi/2} cos^{10}\ x\ dx$

Solution:

Given $\int_0^{\pi/2} cos^{10}\ x\ dx$

We know that $\int_0^{\pi/2} cos^n\, x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{1}{2} \cdot \frac{\pi}{2},\ when\, n\, is\, even\ \cdots$ (1)

Put n = 10 in equation (1)

$$\int_0^{\pi/2} cos^n\, x\, dx = \left(\frac{10-1}{10}\right)\left(\frac{10-3}{10-2}\right)\left(\frac{10-5}{10-4}\right)\left(\frac{10-7}{10-6}\right)\left(\frac{10-9}{10}\right)\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{9}{10}\right)\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) \quad = \frac{63}{512}\pi$$

(v) Evaluate $\int_0^{\pi} sin^2 x\, dx$

Solution:

Given $\int_0^{\pi} sin^2 x\, dx$

$$\int_0^{\pi} sin^2 x\, dx = \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi}(1 - cos\, 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{0}\right)\right]$$

$$= \frac{1}{2}(\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

(vi) Evaluate $\int_0^{\pi/2} sin^{2n+1}\, x\, dx$

Solution:

Given $\int_0^{\pi/2} sin^{2n+1}\, x\, dx$

We know that $\int_0^{\pi/2} sin^n\, x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{2}{3} . 1,\ when\, n\, is\, odd\ \cdots$ (1)

Put n = 2 n + 1 in equation (1)

$$\int_0^{\pi/2} sin^n\, x\, dx = \frac{(2n+1)-1}{2n+1} \cdot \frac{(2n+1)-3}{(2n+1)-2} \cdot \frac{(2n+1)-5}{(2n+1)-4} \cdots 1$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

(vii) Evaluate $\int tan^2\, x\, dx$

Solution:

Given $\int tan^2\, x\, dx$

$$\int tan^2\, x\, dx = \int (sec^2 x - 1)\, dx$$

$$= \int sec^2 x\, dx - \int dx$$

$$= tanx - x + C$$

(viii)Evaluate $\int \boldsymbol{tan^3\, x\, dx}$

Solution:

Given $\int tan^3\, x\, dx$

$$\int tan^3\, x\, dx = \int tan^2\, x\;\; tanx\, dx$$

$$= \int (sec^2x - 1)\, tanx\, dx$$

$$= \int sec^2x\, tanx\, dx - \int tanx\, dx$$

$$= \int tanx\, d(tanx) - \int tanx\, dx$$

$$= \frac{tan^2x}{2} - \log secx + C$$

(ix)Evaluate $\int_{\pi/6}^{\pi/2} \boldsymbol{cot^2\, x\, dx}$

Solution:

Given $\int_{\pi/6}^{\pi/2} cot^2\, x\, dx$

$$\int_{\pi/6}^{\pi/2} cot^2\, x\, dx = \int_{\pi/6}^{\pi/2} (cosec^2\, x - 1)\, dx$$

$$= \int_{\pi/6}^{\pi/2} cosec^2\, x\, dx - \int_{\pi/6}^{\pi/2} dx$$

$$= [-\cot x]_{\pi/6}^{\pi/2} - [x]_{\pi/6}^{\pi/2}$$

$$= (-0) - \left(-\sqrt{3}\right) - \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{1}{3}\pi$$