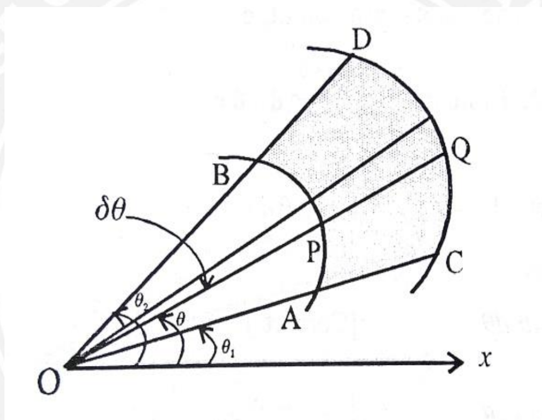
5.2 Double integration in Polar co-ordinates

Consider the integral

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)dr\, d\theta$$

which is in polar form. This integral is bounded over the region by the straight line $\theta = \theta_1, \theta = \theta_2$ and the curves $r = r_1, r = r_2$.To evaluate the integral, we first integrate with respect to r between the limits $r = r_1$ and $r = r_2$(treating θ as a constant).The resulting expression is then integrated with respect to θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.



Geometrically, AB and CD are the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$ so that ABCD is the region of integration. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is performed along PQ (i.e., r varies , θ constant) and the integration with respect to θ

$$\int_{\theta_1}^{\theta_2} f(r,\theta)d\theta$$

means rotation of the strip PQ from AC to BD

Problems based on double integration in Polar Co-ordinates

Example:

Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

Solution:

Given $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

$$= \int_0^{\pi/2}\int_0^{sin\theta} rdrd\theta \quad \text{(Correct form)}$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta = \int_0^{\pi/2}\left[\frac{(sin\theta)^2}{2} - 0\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} sin^2\,\theta d\theta$$

$$= \frac{1}{2}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{8}$$

Example:

Evaluate $\int_0^{\pi}\int_0^{sin\theta} rdrd\theta$

Solution:

Given $\int_0^{\pi}\int_0^{sin\theta} rdrd\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta$$

$$= \int_0^{\pi}\frac{sin^2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{1-cos2\theta}{2}\right] d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{sin2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{\pi}{4}$$

Example:

Evaluate $\int_0^{\pi}\int_0^{a} rdrd\theta$

Solution:

Given $\int_0^{\pi}\int_0^{a} rdrd\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a} d\theta$$

$$= \int_0^{\pi}\frac{a^2}{2} d\theta$$

$$= \frac{a^2}{2}[\theta]_0^{\pi} = \frac{\pi a^2}{2}$$

Example:

Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2cos\theta} r^2 d\theta dr$

Solution:

Given $\int_{-\pi/2}^{\pi/2}\int_{0}^{2cos\theta} r^2 d\theta dr$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{2cos\theta} r^2 dr d\theta \qquad \text{(correct form)}$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2cos\theta} d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{(2cos\theta)^3}{3}-0\right] d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} cos^3\theta\, d\theta$$

$$= \frac{8}{3}(2)\int_{0}^{\pi/2} cos^3\theta\, d\theta$$

$$= \frac{16}{3}\left[\frac{2}{3}.1\right] \qquad = \frac{32}{9}$$

Example:

Evaluate $\int_{0}^{\pi/2}\int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

Solution:

Given $\int_{0}^{\pi/2}\int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

$$= \int_{0}^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-cos\theta)}^{a} d\theta$$

$$= \int_{0}^{\pi/2}\left[\frac{a^3}{3}-\frac{a^3(1-cos\theta)^3}{3}\right] d\theta$$

$$= \frac{a^3}{3}\int_{0}^{\pi/2}[1-(1-cos\theta)^3]\mathrm{d}\theta$$

$$= \frac{a^3}{3}\int_{0}^{\pi/2}[1-(1-3cos\theta+3cos^2\theta-cos^3\theta)]\mathrm{d}\theta$$

$$= \frac{a^3}{3}\int_{0}^{\pi/2}[3cos\theta+3cos^2\theta-cos^3\theta)]d\theta$$

$$= \frac{a^3}{3}\left[(3sin\theta)_0^{\pi/2} - 3\frac{1}{2}\frac{\pi}{2}+\frac{2}{3}\right]$$

$$= \frac{a^3}{3}\left[3-3\frac{\pi}{2}+\frac{2}{3}\right]$$

$$= \frac{a^3}{3}\left[\frac{36-9\pi+8}{12}\right]$$

$$= \frac{a^3}{36}[44 - 9\pi]$$

Exercise

Evaluate the following integrals

1. $\int_0^{\pi/2}\int_{acos\theta}^{a} r^4 drd\theta$ **Ans:** $\left(\pi - \frac{16}{15}\right)\frac{a^5}{10}$

2. $\int_0^{2\pi}\int_{asin\theta}^{a} rdrd\theta$ **Ans:** $\frac{\pi a^2}{4}$

3. $\int_{-\pi/4}^{\pi/4}\int_0^{cos2\theta} rdrd\theta$ **Ans:** $\frac{\pi}{8}$

4. $\int_0^{\pi/2}\int_0^{acos\theta} r\sqrt{a^2-r^2}drd\theta$ **Ans:** $\frac{a^3}{18}(3\pi - 4)$

5. $\int_0^{\pi/4}\int_0^{asin\theta} \frac{r}{\sqrt{a^2-r^2}}drd\theta$ **Ans:** $\frac{a(\pi-3)}{6}$

6. $\int_{b/2}^{b}\int_0^{\pi/2} rd\theta dr$ **Ans:** $\frac{3\pi b^2}{16}$

Area enclosed by plane curves (Cartesian coordinates)

Area = $\int\int dy\,dx$ **(or) Area =** $\int\int dxdy$

Example:

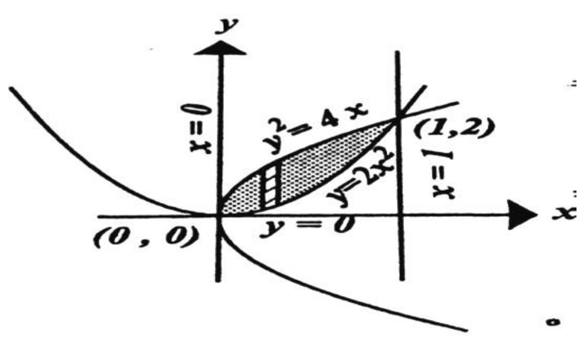
Find the area enclosed by the curves y=$2x^2$ and $y^2 = 4x$

Solution:

Area = $\int\int dy\,dx$

$y: 2x^2 \to 2\sqrt{x}$

$x: 0 \to 1$



Area $= \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\, dx$

$= \int_0^1 [y]_{2x^2}^{2\sqrt{x}} dx$

$= \int_0^1 (2\sqrt{x - 2x^2}) dx$

$= \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$

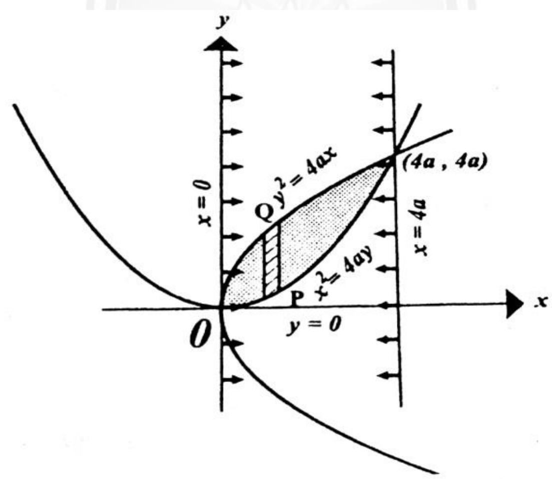
$= \left[\frac{4x^{3/2}}{3} - \frac{2x^3}{3}\right]_0^1$

$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

Example

Find the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$

Solution:



Area $= \int \int dy\, dx$

$y : \frac{x^2}{4a} \to 2\sqrt{ax}$

$x : 0 \to 4a$

$= \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx$

$= \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$

$= \int_0^{4a} (2\sqrt{ax} - \frac{x^2}{4a})\, dx$

$$= \left[\frac{2\sqrt{a\ \ x^{3/2}}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\ (4a)^{3/2} - \frac{(4a)^3}{12a}$$

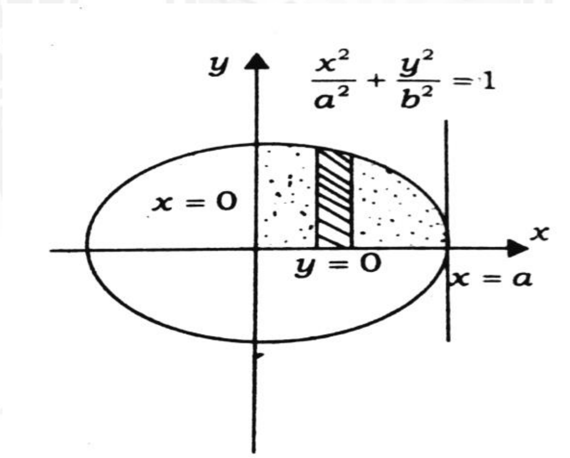
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Example:

Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:



Area =4 $\iint$ dx dy

$x : 0 \rightarrow \frac{a}{b}\sqrt{b^2 - y^2}$

$y : 0 \rightarrow ab$

$$\text{Area} = 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy\, dx$$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}}\ dy$$

$$= 4\int_0^b \left[\frac{a}{b}\sqrt{b^2 - y^2} - 0\right] dy$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}sin^{-1}\left(\frac{y}{b}\right) + \frac{y}{2}\sqrt{b^2 - y^2}\ \right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b^2}{2}\ \frac{\pi}{2} + 0\right) - 0\right]$$

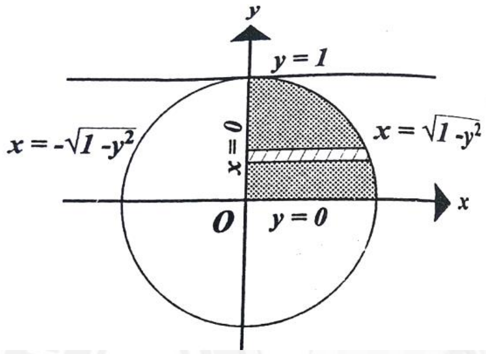
$$= \frac{4ab}{b}\,\frac{b^2}{2}\,\frac{\pi}{2}$$

$$= \pi ab$$

Example:

Evaluate $\iint xy\ dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$

Solution:



$x : 0 \rightarrow \sqrt{1 - y^2}$

$y :\ 0 \rightarrow\ 1$

$$\iint xy\ dx\,dy\ = \int_0^1 \int_0^{\sqrt{1-y^2}} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{2}\int_0^1 (\sqrt{1-y^2})^2 y\,dy$$

$$= \frac{1}{2}\int_0^1 (1-y^2)\,y\,dy$$

$$= \frac{1}{2}\int_0^1 (y - \ y^3)\,dy$$

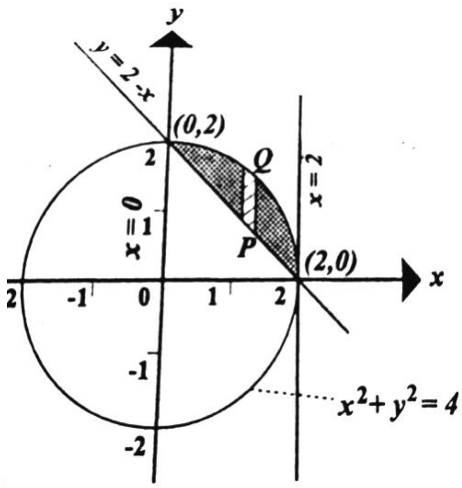
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right]\ =\ \frac{1}{2}\left[\frac{1}{4}\right]\ =\ \frac{1}{8}$$

Example:

Find the smaller of the area bounded by $y = 2 - x$ and $x^2 + y^2 = 4$

Solution:



Area= $\iint dy\,dx$

$y : 2 - x \to \sqrt{4 - x^2}$

$x : 0 \to 2$

$$= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 [\sqrt{4 - x^2} - (2 - x)]dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + \frac{4}{2}\left(\frac{\pi}{2}\right) - 4 + \frac{4^2}{2}$$

$$= \pi - 2 \text{ square unit}$$

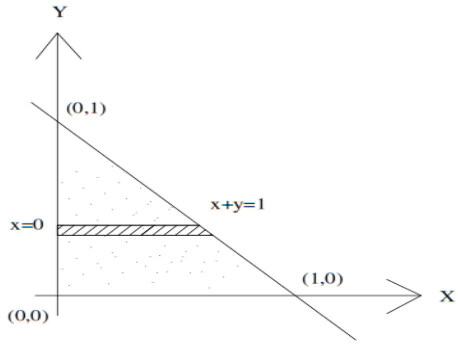
Example:

Evaluate $\iint xy\,dxdy$ over the positive quadrant for which $x + y \leq 1$

Solution:

$x : 0 \to 1 - y$

$y : 0 \to 1$



$$\iint xy\ dxdy = \int_0^1 \int_0^{1-y} xy\ dx\ dy$$

$$= \int_0^1 \left(\frac{x^2 y}{2}\right)_0^{1-y} dy$$

$$= \int_0^1 \frac{1-y^2 y}{2} dy$$

$$= \frac{1}{2}\int_0^1 (y^2 - 2y^2 + y^3) dy$$

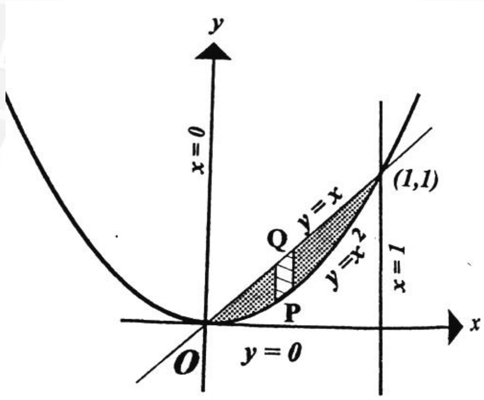
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] \quad = \frac{1}{2}\left[\frac{16-8+3}{12}\right] \quad = \frac{1}{24}$$

Example:

Using double integral find the area bounded by y = x and y = x^2

Solution



Area $= \iint dy\ dx$

$y :\ x^2 \to x$

$x :\ 0 \to\ 1$

$$= \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx$$

$$= \int_0^1 (x - x^2) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

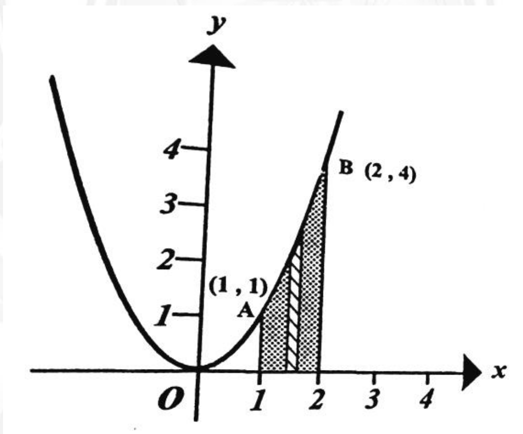
Example:

Evaluate $\iint (x^2 + y^2)dxdy$ where A is area bounded by the curves x^2=y, x=1 and x=2 about x axis

Solution:

$y : 0 \to x^2$

$x : 1 \to 2$



$$\iint (x^2 + y^2)dxdy = \int_1^2 \int_0^{x^2} (x^2 + y^2)dydx$$

$$= \int_1^2 \left[x^2 y + \frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_1^2 (x^4 + \frac{x^6}{3})dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{21}\right]_1^2$$

$$= [\frac{2^5}{5} + \frac{2^7}{21} - \frac{1}{5} - \frac{1}{21}]$$

$$= \frac{1286}{105}$$

Example:

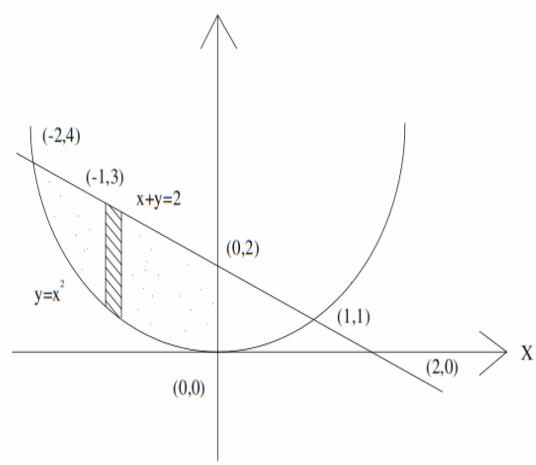
Find the area enclosed by the curves $y = x^2$ and $x + y - 2 = 0$

Solution:

Given $y=x^2$ and $x + y - 2 = 0$

x	0	1	2	-1	-2
Y=2$-x$	2	1	0	3	4

x	0	1	-1	2	-2
$y = x^2$	0	1	1	4	4



Area = $\iint$ dy dx

$y: x^2 \to 2 - x$

$x: -2 \to 1$

$$\int_{-2}^{1}\int_{x^2}^{2-x} dy\, dx = \int_{-2}^{1}[y]_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1}(2 - x - x^2)dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left[2 - \frac{1}{2} - \frac{1}{3}\right] - \left[-4 - \frac{4}{2} + \frac{8}{3}\right] = \frac{27}{6}$$

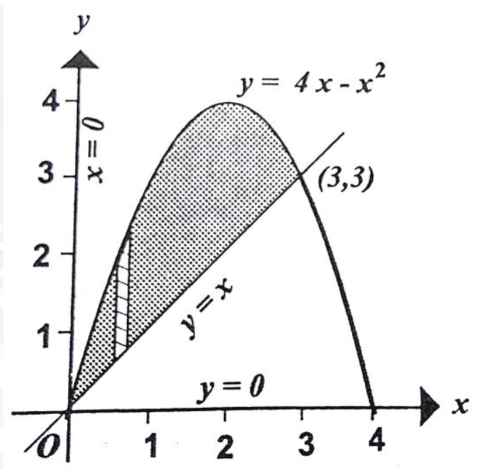
Example:

Find by double integration the area lying between the parabola y=4x-x^2 and the line y=x

Solution:

Given $y = 4x - x^2$ and $y = x$

x	0	1	2	-1	-2	3
$y = 4x - x^2$	0	3	4	-5	-12	3



$Area = \iint dy\, dx$

$y : x \to 4x - x^2$

$x : 0 \to 3$

$$\int_0^3 \int_x^{4x-x^2} dy\, dx = \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (4x - x^2 - x) dx$$

$$= \int_0^3 (3x - x^2) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}$$

Exercise:

1. Evaluate $\iint xdy\,dx$ over the region between the parabola $y^2 = x$ and the lines $x + y = 2$, $x = 0$ and $x = 1$ **Ans:** $\frac{4}{15}$

2. Evaluate $\iint y^2\,dxdy$ over the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **Ans:** $\frac{\pi ab^3}{4}$

3. Find the area between the curve $y^2 = 4 - x$ and the line $y^2 = x$ **Ans:** $\frac{16\sqrt{2}}{3}$

Area Enclosed by Plane Curves [Polar co-ordinates]

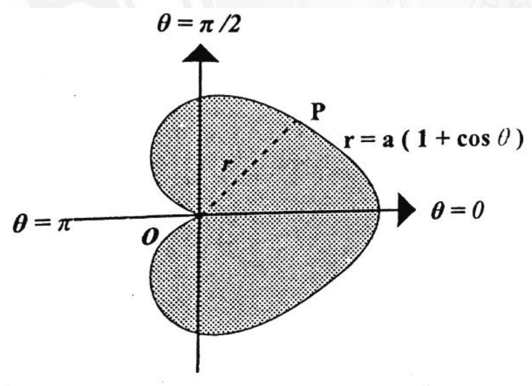
Area= $\iint r\,dr\,d\theta$

Problems Based on Area Enclosed by Plane Curves Polar Coordinates

Example:

Find using double integral, the area of the cardioid $r = a(1 + \cos\theta)$

Solution:



Area= $\iint r\,dr\,d\theta$

The curve is symmetrical about the initial line

θ varies from : $0 \to \pi$

r varies from: $0 \to a(1 + \cos\theta)$

Hence, required area $= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a(1+\cos\theta)} r\,dr\,d\theta$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)} d\theta \quad = \int_0^{\pi}[r^2]_{r=0}^{r=a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi}[a^2(1+\cos\theta)^2 - 0]d\theta = a^2\int_0^{\pi}[1 + \cos^2\theta + 2\cos\theta]d\theta$$

$= a^2 \int_0^{\pi} \left[1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$

$= \frac{a^2}{2} \int_0^{\pi} [2 + 1 + \cos 2\theta + 4\cos\theta]\, d\theta = \frac{a^2}{2} \int_0^{\pi} [3 + \cos 2\theta + 4\cos\theta]\, d\theta$

$= \frac{a^2}{2} \left[3\theta + \frac{\sin 2\theta}{2} + 4\sin\theta\right]_0^{\pi} = \frac{a^2}{2} [(3\pi + 0 + 0) - (0 + 0 + 0)]$

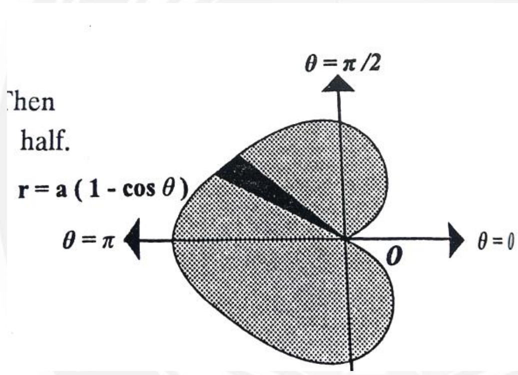
$= \frac{3}{2} a^2 \pi$ square units.

Example:

Find the area of the cardioid $r = a(1 - \cos\theta)$

Solution:

Area= $\iint r dr d\theta$



The curve is symmetrical about the initial line.

θ varies from : $0 \to \pi$

r varies from: $0 \to a(1 - \cos\theta)$

Hence, required area $= 2 \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1-\cos\theta)} r dr d\theta$

$= 2 \int_0^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta = \int_0^{\pi} [r^2]_{r=0}^{r=a(1-\cos\theta)} d\theta$

$= \int_0^{\pi} [a^2(1 - \cos\theta)^2 - 0] d\theta = a^2 \int_0^{\pi} [1 + \cos^2\theta - 2\cos\theta] d\theta$

$= a^2 \int_0^{\pi} \left[1 + \frac{1+\cos 2\theta}{2} - 2\cos\theta\right] d\theta$

$= \frac{a^2}{2} \int_0^{\pi} [2 + 1 + \cos 2\theta - 4\cos\theta]\, d\theta = \frac{a^2}{2} \int_0^{\pi} [3 + \cos 2\theta - 4\cos\theta]\, d\theta$

$= \frac{a^2}{2} \left[3\theta + \frac{\sin 2\theta}{2} - 4\sin\theta\right]_0^{\pi} = \frac{a^2}{2} [(3\pi + 0 - 0) - (0 + 0 - 0)]$

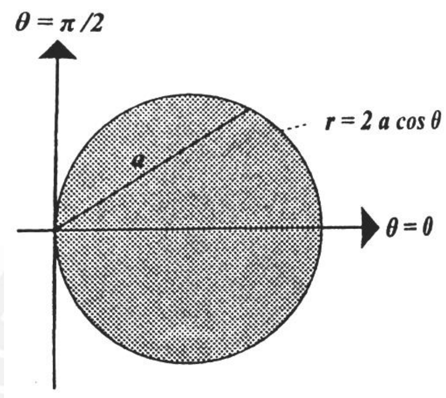
$= \frac{3}{2} a^2 \pi$ square units.

Example:

Find the area of a circle of radius 'a' by double integration in polar co-ordinates.

Solution:

Area= $\iint \mathbf{rdrd\theta}$



The equation of circle with pole on the circle and diameter through the point as initial line is $r = 2a\cos\theta$

Area = 2 × upper area

$$= 2\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2a\cos\theta} rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} [r^2]_0^{2a\cos\theta} d\theta$$

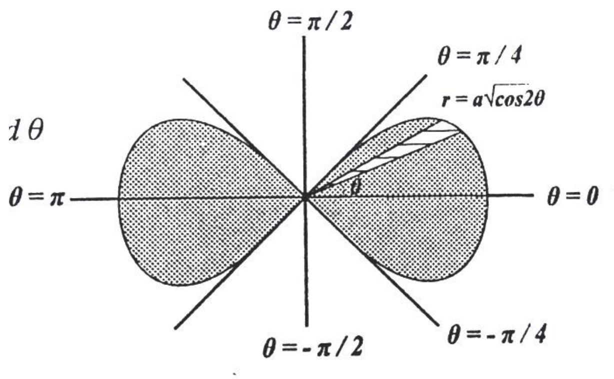
$$= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \quad = 4a^2 . \frac{1}{2} . \frac{\pi}{2} = \pi a^2 \text{ square units.}$$

Example:

Find the area of the lemniscates $\mathbf{r^2 = a^2\cos 2\theta}$ by double integration. [A.U R-08]

Solution:

Area= $\iint rdrd\theta$



Area = 4 × area of upper half of one loop.

$$= 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}(r^2)_0^{a\sqrt{\cos 2\theta}}d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= 2a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}}$$

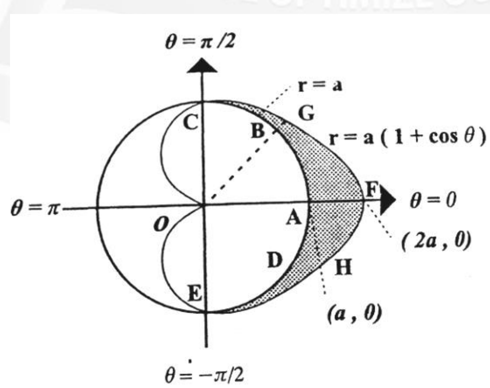
$$= a^2 \text{ square units.}$$

Example:

Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$, by double integration. [A.U 2014]

Solution:

Area= $\iint r\,dr\,d\theta$



Both the curves are symmetric about the initial line.

Hence, the required area = $2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=a}^{r=a(1+cos\theta)} rdrd\theta$

$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{r=a}^{r=a(1+cos\theta)} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{a^2(1+cos\theta)^2}{2}-\frac{a^2}{2}\right] d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}[1+cos^2\theta+2cos\,\theta-1]d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}\left[\frac{1+cos2\theta}{2}+2cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[1+cos2\theta+4cos\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[\theta+\frac{sin2\theta}{2}+4sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\left(\frac{\pi}{2}+0+4\right)-(0+0+0)\right]$$

$$= \frac{a^2}{4}(\pi+8)\text{square units.}$$

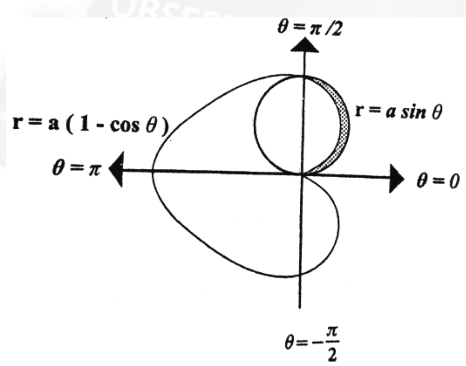
.

Example:

Find the area inside the circle $\mathbf{r=asin\theta}$ and outside the cardioid $\boldsymbol{r=a(1-cos\theta)}$

[A.U.Jan.2009]

Solution:



From the figure, we get

θ varies from : $0 \to \frac{\pi}{2}$

r varies from: $a(1-cos\theta) \to asin\theta$

The required area = $\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=a(1-\cos\theta)}^{r=a\sin\theta)} r dr d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{r=a(1-cos\theta)}^{r=asin\theta)} d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{a^2 sin^2\theta}{2} - \frac{a^2(1-cos\theta)^2}{2}\right] d\theta$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[sin^2\theta - 1 - cos^2\theta + 2cos\theta]\, d\theta$

$= \frac{a^2}{2}\left[\int_0^{\frac{\pi}{2}} sin^2\theta d\theta - \int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} cos^2\theta d\theta + 2\int_0^{\frac{\pi}{2}} cos\theta d\theta\right]$

$= \frac{a^2}{2}\left[-[\theta]_0^{\frac{\pi}{2}} + 2[sin\theta]_0^{\frac{\pi}{2}}\right]$ $\therefore \int_0^{\frac{\pi}{2}} sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$

$= \frac{a^2}{2}\left[-\left(\frac{\pi}{2} - 0\right) + 2(1-0)\right]$ $= \frac{1}{2}\frac{\pi}{2}$

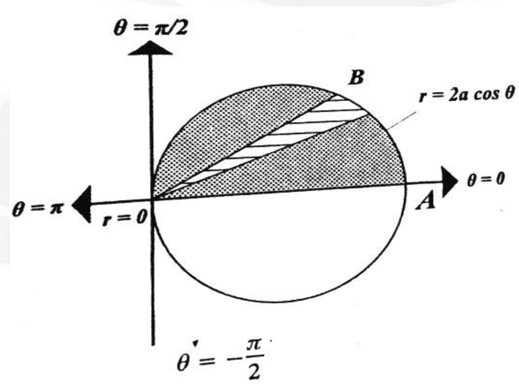
$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$

$= \frac{a^2}{4}(4-\pi)$ square units.

Example:

Evaluate $\int_R \int r^2 \sin\theta\, dr d\theta$ where R is the semi circle $r = 2a\cos\theta$ above the initial line.

Solution:



θ varies from : $0 \to \frac{\pi}{2}$

r varies from: $0 \to 2a\cos\theta$

Let $I = \int\int r^2 \sin\theta\, dr d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2acos\theta} (r^2 sin\theta) dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\ sin\theta\right]_{r=0}^{r=2acos\theta)}\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{8a^3}{3} cos^3\theta\ sin\theta - 0\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{8a^3}{3} cos^3\theta\ sin\theta\ \ d\theta$$

$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} cos^3\theta\ sin\theta\ \ d\theta$$

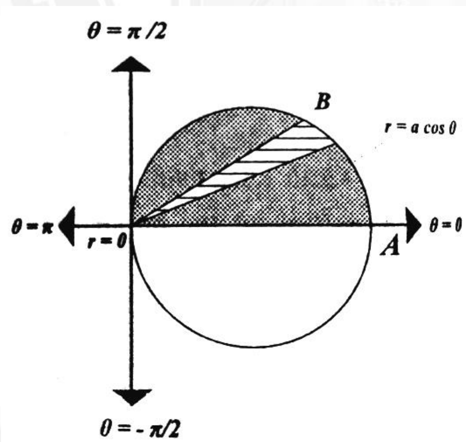
$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} cos^3\theta\ \ d(cos\theta) = \frac{8a^3}{3} \left[\frac{cos^4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8a^3}{3} \left[0 - \frac{1}{4}\right] = \frac{-2a^3}{3}$$

Example:

Evaluate $\int\int r\sqrt{a^2 - r^2}\ dr d\theta$ over the upper half of the circle $r = acos\theta$.

Solution:



$\theta\ \ varies\ from\ : 0 \to \frac{\pi}{2}$

$r\ varies\ from: 0 \to acos\theta$

$Let\ \ \ I = \int\int r\sqrt{a^2 - r^2}\ dr d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^{acos\theta} r\sqrt{a^2 - r^2}\ \ dr d\theta$$

Put $a^2 - r^2 = t^2$

$-2rdr = 2tdt$

$-r\,dr = t\,dt$

$r \to 0 \Rightarrow t \to a$

$r \to a\cos\theta \Rightarrow t \to a\sin\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \sqrt{t^2}\ (-tdt)d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} t^2 dt d\theta \quad = \quad = -\int_0^{\frac{\pi}{2}} \left[\frac{t^3}{3}\right]_{r=a}^{r=a\sin\theta)} d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} [t^3]_a^{a\sin\theta}\ d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} [a^3\sin^3\theta - a^3]\ d\theta$$

$$= -\frac{a^3}{3} \int_0^{\frac{\pi}{2}} [\sin^3\theta - 1]\ d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} [1 - \sin^3\theta]\ d\theta$$

$$= \frac{a^3}{3} [\theta]_0^{\frac{\pi}{2}} - \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \sin^3\theta\ d\theta$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2} - 0\right] - \frac{a^3}{3}\left[\frac{2}{3}.1\right] \qquad \left[\therefore \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = \frac{2}{3}.1\right]$$

$$= \frac{a^3}{3}\frac{\pi}{2} - \frac{a^3}{3}\frac{2}{3}$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

Exercise:

1. Evaluate $\iint r\sin\theta\, drd\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

Ans: $\frac{4a^2}{3}$

2. Evaluate $\iint \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscates $r^2 = a^2\cos 2\theta$

Ans: $a\left(2-\frac{\pi}{2}\right)$

3. Find by double integration the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Ans: 3π

4. Find the area outside $r = 2a\cos\theta$ and inside $r = a(1+\cos\theta)$ **Ans**: $\frac{\pi a^2}{2}$