3.5 Domain and Range of Functions

In mathematics, the **domain** and **range** are fundamental concepts related to functions. Understanding them is essential for analyzing and interpreting the behavior of mathematical functions.

Function Definition

A **function** is a relation that assigns exactly one output for every input from a given set. If we have a function f:X→Yf: X \to Yf:X→Y, it means that for each element x∈Xx \in Xx∈X (the domain), there is exactly one element f(x)∈Yf(x) \in Yf(x)∈Y (the codomain).

- **Domain**: The set of all possible inputs (values) for which the function is defined.
- **Range**: The set of all possible outputs (values) that the function can produce.

Domain of a Function

The **domain** of a function refers to the set of all possible input values for which the function is defined. To determine the domain, we typically follow these steps:

1. **Identify restrictions** on the variable (input) such as:
    - Division by zero (denominators cannot be zero).
    - Square roots (the expression inside the root must be non-negative if considering real numbers).
    - Logarithms (the argument of the logarithm must be positive).
2. **Look for context**:
    - Real-world applications may impose additional restrictions (e.g., negative values of time or distance may not be meaningful).

Examples of Domain Determination:

- **Rational functions** (fractions):
  For f(x)=1x−2f(x) = \frac{1}{x-2}f(x)=x−21, the domain excludes x=2x = 2x=2, because division by zero is undefined.
    - Domain: x∈R∖{2}x \in \mathbb{R} \setminus \{2\}x∈R∖{2} (all real numbers except 2).
- **Square root functions**:
  For g(x)=x−1g(x) = \sqrt{x-1}g(x)=x−1, the expression inside the square root must be non-negative, so x−1≥0x - 1 \geq 0x−1≥0, which gives x≥1x \geq 1x≥1.
    - Domain: x∈[1,∞)x \in [1, \infty)x∈[1,∞).
- **Logarithmic functions**:
  For h(x)=log(x−3)h(x) = \log(x-3)h(x)=log(x−3), the argument of the logarithm must be positive, so x−3>0x - 3 > 0x−3>0, which gives x>3x > 3x>3.
    - Domain: x∈(3,∞)x \in (3, \infty)x∈(3,∞).

Range of a Function

The **range** of a function refers to the set of all possible output values that the function can produce, given its domain.

Steps to find the range:

1. **Look at the function**: Check how the output behaves based on the input.
2. **Analyze the behavior**:
   - For polynomial functions, consider end behavior (e.g., as $x \to \infty$).
   - For rational functions, consider asymptotes or any restrictions.
   - For trigonometric functions, remember their periodicity and amplitude.

Examples of Range Determination:

- **Quadratic function**:
  For $f(x) = x^2$, the output is always non-negative since $x^2 \geq 0$.
  - Range: $f(x) \in [0, \infty)$.
- **Rational function**:
  For $g(x) = \frac{1}{x}$, the output can take any real value except 0, because the function approaches but never equals zero.
  - Range: $g(x) \in \mathbb{R} \setminus \{0\}$.
- **Trigonometric function**:
  For $\sin(x)$, the values oscillate between -1 and 1.
  - Range: $\sin(x) \in [-1, 1]$.

Domain and Range in Different Types of Functions

Polynomial Functions:

- The domain of any polynomial function is always $\mathbb{R}$ (all real numbers).
- The range depends on the degree and leading coefficient:
  - If the degree is even (e.g., $f(x) = x^2$), the range is $[0, \infty)$ (for non-negative outputs).
  - If the degree is odd (e.g., $f(x) = x^3$), the range is $\mathbb{R}$ (all real numbers).

Rational Functions:

- The domain excludes values that make the denominator zero.
- The range depends on the behavior of the function and possible asymptotes.
  - Example: $f(x) = \frac{1}{x}$ has domain $\mathbb{R} \setminus \{0\}$ and range $\mathbb{R} \setminus \{0\}$.

Square Root Functions:

- The domain is restricted to values where the expression inside the square root is non-negative.
- The range is typically non-negative for real functions.
  - Example: $f(x) = \sqrt{x-3}$ has domain $x \geq 3$ and range $[0, \infty)$.

Logarithmic Functions:

- The domain requires that the argument inside the logarithm be positive.
- The range is typically $\mathbb{R}$.
  - Example: $f(x) = \log(x-2)$ has domain $x > 2$ and range $\mathbb{R}$.

Trigonometric Functions:

- For functions like $\sin(x)$ and $\cos(x)$, the domain is $\mathbb{R}$ (all real numbers), and the range is typically $[-1, 1]$.
- For functions like $\tan(x)$, the domain excludes values where the denominator (cosine) equals zero, and the range is $\mathbb{R}$.

How to Express Domain and Range

- **Interval notation**:
  - $[a, b]$ means all values from a to b inclusive.
  - (a, b) means all values from a to b exclusive.
  - $(-\infty, a)$ means all values less than a.
  - (a, ∞) means all values greater than a.
- **Set-builder notation**:
  - Domain: $\{ x \mid x \geq 3 \}$.
  - Range: $\{ y \mid y \geq 0 \}$.

Special Cases

- **Piecewise Functions**:
  A function defined in pieces may have different domains and ranges in each section. For example:

  $$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

  - Domain: $\mathbb{R}$
  - Range: $(-\infty, 0] \cup [0, \infty)$.
- **Implicit Functions**:
  For equations like $x^2 + y^2 = 1$ (the equation of a circle), the domain and

range may need to be determined by solving for yyy in terms of xxx, or considering geometric interpretations.

Example Problem:

Given f(x)=2x+1x2−4f(x) = \frac{2x+1}{x^2 - 4}f(x)=x2−42x+1, find the domain and range.

1. **Domain**: The denominator x2−4=0x^2 - 4 = 0x2−4=0 when x=2x = 2x=2 or x=−2x = -2x=−2, so the domain is all real numbers except x=2x = 2x=2 and x=−2x = -2x=−2.
   - Domain: x∈R∖{−2,2}x \in \mathbb{R} \setminus \{-2, 2\}x∈R∖{−2,2}.
2. **Range**: Analyze the behavior of the function as x→∞x \to \inftyx→∞ or near the asymptotes x=2x = 2x=2 and x=−2x = -2x=−2. Through further analysis, we conclude the range is all real numbers except for 0.
   - Range: f(x)∈R∖{0}f(x) \in \mathbb{R} \setminus \{0\}f(x)∈R∖{0}.

Example 1: Linear Function

Function: f(x)=3x+2f(x) = 3x + 2f(x)=3x+2

- **Domain:** Since this is a polynomial (a linear function), there are no restrictions on xxx. You can plug in any real number for xxx, so the domain is all real numbers.

  Domain: (−∞,∞)(-\infty, \infty)(−∞,∞)

- **Range:** As xxx can take any real value, f(x)f(x)f(x) can also take any real value. So, the range is all real numbers.

  Range: (−∞,∞)(-\infty, \infty)(−∞,∞)

Example 2: Rational Function

Function: f(x)=1x−3f(x) = \frac{1}{x-3}f(x)=x−31

- **Domain:** The function has a denominator, and division by zero is undefined. The denominator x−3x - 3x−3 equals zero when x=3x = 3x=3, so x=3x = 3x=3 is not in the domain.

  Domain: (−∞,3)∪(3,∞)(-\infty, 3) \cup (3, \infty)(−∞,3)∪(3,∞)

- **Range:** As the denominator can never be zero and the function approaches infinity as xxx approaches 3, the range is all real numbers except 000. It can't output a value of 0, because for f(x)=0f(x) = 0f(x)=0, the numerator would have to be 0, which never happens.

  Range: (−∞,0)∪(0,∞)(-\infty, 0) \cup (0, \infty)(−∞,0)∪(0,∞)

Example 3: Square Root Function

Function: f(x)=x−4f(x) = \sqrt{x - 4}f(x)=x−4

- **Domain:** The expression inside the square root, x−4x - 4x−4, must be greater than or equal to 0 because the square root of a negative number is not a real number. So, we need:

  x−4≥0⇒x≥4x - 4 \geq 0 \quad \Rightarrow \quad x \geq 4x−4≥0⇒x≥4

  Domain: [4,∞)[4, \infty)[4,∞)

- **Range:** Since the square root function always produces non-negative values, the smallest value for f(x)f(x)f(x) occurs when x=4x = 4x=4, giving f(4)=0=0f(4) = \sqrt{0} = 0f(4)=0=0. As xxx increases, f(x)f(x)f(x) increases as well.

  Range: [0,∞)[0, \infty)[0,∞)

Example 4: Logarithmic Function

Function: f(x)=log(x−1)f(x) = \log(x - 1)f(x)=log(x−1)

- **Domain:** The argument of the logarithm must be positive, so we need:

  x−1>0⇒x>1x - 1 > 0 \quad \Rightarrow \quad x > 1x−1>0⇒x>1

  Domain: (1,∞)(1, \infty)(1,∞)

- **Range:** Logarithmic functions can produce any real number, as they can take any positive input x>1x > 1x>1 and map it to any real output.

  Range: (−∞,∞)(-\infty, \infty)(−∞,∞)

Example 5: Quadratic Function

Function: f(x)=x2−4x+3f(x) = x^2 - 4x + 3f(x)=x2−4x+3

- **Domain:** Since this is a polynomial, there are no restrictions on xxx, so the domain is all real numbers.

  Domain: (−∞,∞)(-\infty, \infty)(−∞,∞)

- **Range:** To find the range, let's first complete the square to rewrite the function in vertex form:

f(x)=(x2−4x+4)−4+3=(x−2)2−1f(x) = (x^2 - 4x + 4) - 4 + 3 = (x - 2)^2 - 1f(x)=(x2−4x+4)−4+3=(x−2)2−1

The vertex form of the function is f(x)=(x−2)2−1f(x) = (x - 2)^2 - 1f(x)=(x−2)2−1, which has a minimum value of −1-1−1 at x=2x = 2x=2. Since the square of any real number is non-negative, (x−2)2≥0(x - 2)^2 \geq 0(x−2)2≥0, so the smallest value of f(x)f(x)f(x) is −1-1−1.

Range: [−1,∞)[-1, \infty)[−1,∞)

Example 6: Piecewise Function

Function:

f(x)={x+2if x≥0−x+1if x<0f(x) = \begin{cases} x + 2 & \text{if } x \geq 0 \\ -x + 1 & \text{if } x < 0 \end{cases}f(x)={x+2−x+1if x≥0if x<0

- **Domain:** The function is defined for all real numbers, since it provides a rule for both x≥0x \geq 0x≥0 and x<0x < 0x<0.

  Domain: (−∞,∞)(-\infty, \infty)(−∞,∞)

- **Range:** For x≥0x \geq 0x≥0, the function f(x)=x+2f(x) = x + 2f(x)=x+2 has a range of [2,∞)[2, \infty)[2,∞). For x<0x < 0x<0, the function f(x)=−x+1f(x) = -x + 1f(x)=−x+1 has a range of (−∞,1)(-\infty, 1)(−∞,1). Since the function includes all values from 222 and higher, and all values from negative infinity up to 111, the full range is:

  Range: (−∞,1)∪[2,∞)(-\infty, 1) \cup [2, \infty)(−∞,1)∪[2,∞)

Recap of Steps for Finding Domain and Range:

1. **Domain:**
   - Identify any values of xxx that make the function undefined (e.g., division by zero, negative square roots, logarithms of non-positive numbers).
   - Exclude those values from the domain.
2. **Range:**
   - Consider the type of function and the behavior of the function (e.g., polynomial, rational, square root, etc.).
   - Look for maximum or minimum values, and how the function behaves as xxx approaches extreme values.