Boolean Algebra:

A complemented distributive lattice is called Boolean Algebra.

A Boolean algebra is distributive lattice with "0" element and "1" element in which every element has a complement.

A Boolean algebra is a non empty set with 2 binary operations $\wedge$ and $\vee$ and is satisfied by the following conditions. $\forall a, b, c \in L$

1. $L_1: a \wedge a = a$ and $a \vee a = a$

2. $L_2: a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

3. $L_3: a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$

4. $L_4: a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

5. $D_1: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

6. $D_2: a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

7. There exist between "0" and "1" such that $a \wedge 0 = 0, a \vee 0 = a, a \wedge 1 = a$ and $a \vee 1 = 1 \forall a$

8. $\forall a \in L$, there exist corresponding element a' in L such that $a \wedge a' = 0$ and $a \vee a' = 1$

Note:

Boolean Sum is defined as $1 + 1 = 1,\ 1 + 0 = 1,\ 0 + 1 = 1,\ 0 + 0 = 0$

Boolean Product is defined as $1 \cdot 1 = 1,\ 1 \cdot 0 = 0,\ 0 \cdot 1 = 0,\ 0 \cdot 0 = 0$

Absorption law in Boolean Algebra

1. Prove that $\boldsymbol{a + ab = a}$

Solution:

LHS $= a + ab$

$= a(1 + b)$ (Distributive law)

$= a(1)$ $(1 + a) = 1$

$a + ab = a$ $(a \cdot 1 = a)$

2. Prove that $\boldsymbol{a + \bar{a}b = a + b}$

Solution:

LHS $= a + \bar{a}b$

$= a + ab + \bar{a}b$ $(a = a + ab)$

$= a + b(a + \bar{a})$ (Distributive law)

$= a + b(1)$ $(a + \bar{a}) = 1$ $(a \cdot 1 = a)$

= RHS

3. Prove that $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{c})=\boldsymbol{a}+\boldsymbol{bc}$

Solution:

LHS = $(a+b)(a+c)$

$= aa + ac + ab + bc$ (Distributive law)

$= a + ac + ab + bc$ $(a\cdot a = a)$

$= a(1+c) + ab + bc$ (Distributive law)

$= a + ab + bc$ $(1 + a = 1)$

$= a + bc$ $(a + ab = a)$

= RHS

4. In any Boolean Algebra, show that $\boldsymbol{a}=\boldsymbol{b} \Leftrightarrow \boldsymbol{a}\overline{\boldsymbol{b}}+\overline{\boldsymbol{a}}\boldsymbol{b}=\mathbf{0}$

Proof:

Let $(B,\cdot,+,0,1)$ be any Boolean Algebra.

Let $a,b \in B\ an\ d\ a=b$ . . . (1)

Claim: $a\bar{b}+\bar{a}b=0$

Now $a\bar{b}+\bar{a}b = a\cdot\bar{b}+\bar{a}.b$

$= a \cdot \bar{a} + \bar{a}.a$ using (1)

$= 0 + 0$ (since $a \cdot \bar{a} = 0$)

$= 0$

Conversely, assume $a\bar{b} + \bar{a}b = 0$

$\Rightarrow a + a\bar{b} + \bar{a}b = a$ (Left Cancellation law)

$\Rightarrow a + a\bar{b} = a$ (Absorption law)

$\Rightarrow (a + \bar{a}) \cdot (a + b) = a$ (Distributive law)

$\Rightarrow 1 \cdot (a + b) = a$ $(a + \bar{a} = 1)$

$\Rightarrow (a + b) = a$ $(a \cdot 1 = a)$ . . . (a)

Consider $a\bar{b} + \bar{a}b = 0$

$\Rightarrow a\bar{b} + \bar{a}b + b = b$ (Right Cancellation law)

$\Rightarrow a\bar{b} + b = b$ (Absorption law)

$\Rightarrow (a + b) \cdot \left(b + \bar{b}\right) = b$ (Distributive law)

$\Rightarrow (a + b) \cdot 1 = a$ $(b + \bar{b} = 1)$

$\Rightarrow (a + b) = b$ $(b \cdot 1 = b)$ . . . (b)

From (a) and (b) we get $a = a + b = b$

Hence $a = b$

5. If a and b are two elements of a Boolean algebra, then prove that

$$\boldsymbol{a + (a \cdot b) = a, a \cdot (a + b) = a}$$

Proof:

Consider $a + (a \cdot b) = a = a \cdot 1 + (a \cdot b)$

$= a \cdot (1 + b)$

$= a \cdot 1 \qquad [a + 1 = 1, 1 + a = 1]$

$= a$

Consider $a \cdot (a + b) = a = a \cdot a + (a \cdot b)$

$= a + (a \cdot b)$

$= a \cdot 1 + a \cdot b$

$= a \cdot (1 + b)$

$= a \cdot 1 \qquad [a \cdot a = a, a \cdot 0 = 0]$

$= a$

Hence the proof.

6. Prove that in a Boolean algebra, the complement of any element is unique.

Proof:

Let b and c be the complements of the element a.

Then $b + a = 1,\ b \cdot a = 0$

$a + c = 1,\ a \cdot c = 0$

Consider $b = 1 \cdot b$

$= (a + c) \cdot b$

$= a \cdot b + c \cdot b$

$= 0 + c \cdot b$

$= a \cdot c + c \cdot b$

$= c \cdot (a + b)$

$= 1 \cdot c$

$= c$

Hence the complement is unique.

7. In a Boolean algebra show that the following statements are equivalent. For any a and b (i) $\boldsymbol{a + b = b}$ (ii) $\boldsymbol{a \cdot b = a}$ (iii) $\boldsymbol{a' + b = 1}$ (iv) $\boldsymbol{a \cdot b' = 0}$ (v) $\boldsymbol{a \leq b}$

Proof:

To prove $(i) \Rightarrow (ii)$

Assume that $a + b = b$

To prove that $a \cdot b = a$

Now $a = a \cdot (a + b)$

$= a \cdot b$

Hence $(i) \Rightarrow (ii)$

To prove $(ii) \Rightarrow (iii)$

Assume that $a \cdot b = a$

To prove that $a' + b = 1$

Now $a' + b = (a \cdot b')' + b$

$= a' + b' + b$

$= a' + 1$

$= 1$

Hence $(ii) \Rightarrow (iii)$

To prove $(iii) \Rightarrow (iv)$

Assume that $a' + b = 1$

To prove that $a \cdot b' = 0$

Taking complement on both sides

$\Rightarrow (a' + b)' = 1'$

$\Rightarrow a \cdot b' = 0$

Hence $(iii) \Rightarrow (iv)$

To prove $(iv) \Rightarrow (v)$

Assume that $a \cdot b' = 0$

To prove that $a \leq b$

Then $a \cdot b = a \cdot b + 0$

$= a \cdot b + a \cdot b'$

$= a(b + b')$

$= a \cdot 1$

$= a$

Hence $(iv) \Rightarrow (v)$

To prove $(v) \Rightarrow (i)$

Assume that $a \leq b$

To prove that $a + b = b$

We have $a \cdot b = b$

$\Rightarrow a + b = (a \cdot b) + b$

$= a \cdot b + 1 \cdot b$

$= (a + 1) \cdot b$

$= 1 \cdot b$

$= b$

Hence the proof.

8. Prove that in a Boolean algebra

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a}' + \boldsymbol{c}) = \boldsymbol{ac} + \boldsymbol{a}'\boldsymbol{b} = \boldsymbol{ac} + \boldsymbol{a}'\boldsymbol{b} + \boldsymbol{bc}$$

Proof:

Now, $(a + b) \cdot (a' + c) = (a + b) \cdot a' + (a + b) \cdot c$

$= a' \cdot (a + b) + (a + b) \cdot c$

$= aa' + a'b + ac + bc$

$= 0 + a'b + ac + bc$

$= a'b + ac + bc$

$= ac(b + b') + a'b(c + c') + bc(a + a')$

$= abc + ab'c + a'bc + a'bc' + abc + a'bc$

$= abc + ab'c + a'bc + a'bc'$

$= abc + ab'c + a'b(c + c')$

$= ac(b + b') + a'b(c + c')$

$= ac(1) + a'b(1)$

$= ac + a'b$

= RHS

9. Show that in a Boolean algebra the law of the double complement holds.

(or) Prove the involution law $(\boldsymbol{a}')' = \boldsymbol{a}$

Solution:

It is enough to prove that $a' + a = 1$ and $a \cdot a' = 0$

By domination laws of Boolean algebra, we get

$a' + a = 1$ and $a \cdot a' = 0$

By commutative law, we get $a' + a = 1$ and $a \cdot a' = 0$

Therefore complement of a' is a

$\Rightarrow (a')' = a$

$\Rightarrow a'' = a$

Hence the proof.