UNITV

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

PROBLEMS BASED ON CRANK-NICHOLSON FORMULA

1. **Use Crank Nicholson formula ,Solve** $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$ **,** $0 < x < 1$ and $t > 0$ $given$ $u(x,0) = 0, u(0,t) = 0, u(1,t) = 100t$ **Compute** $u(x,t)$ **for one time step taking** $\Delta x = \frac{1}{4}$

Solution:

By Using method , $k = ah^2 = 16\left(\frac{1}{16}\right) = 1$

Here a=16, $\therefore h = \frac{1}{4}$

We have

$u_{i,j+1} = \frac{1}{4}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j})$……………………..(i)

$\rightarrow x\ Direction$

j \ i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\downarrow t$ 1	0	u_1	u_2	u_2	100

$$u_1 = \frac{1}{4}(0+0+0+u_2) \qquad \therefore u_1 = \frac{u_2}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$

$$u_2 = \frac{1}{4}(u_1+u_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

$$u_3 = \frac{1}{4}(0+0+u_2+100)$$

$$u_3 = \frac{1}{4}(u_2+100) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Substitute u_1, u_3 values in (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}(2u_2+100)\right] = \frac{1}{8}u_2 + \frac{25}{4}$$

$$u_2 = \frac{50}{7} = 7.1429$$

$$u_1 = 1.7857;\ u_3 = 26.7857$$

2. **Use Crank Nicholson formula ,Solve** $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 5$ and $t \geq 0$ $given$ $u(x, 0) = 20, u(0, t) = 0, u(5, t) = 100$ **Compute** $u(x, t)$ **for time step taking** $h = 1$

Solution :

By Using method ,$k = ah^2$ **, h=1**

$$\therefore k = 1$$

$\rightarrow x\ Direction$

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	20
$\downarrow t$ 1	0	u_1	u_2	u_2	u_3	100

We have

$$u_{i,j+1} = \frac{1}{4}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j})$$

$4\, u_{i,j+1} = (u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j})$................(i)

$$4u_1 = (0 + 0 + u_2 + 20)$$

$$4u_1 - u_2 = 20 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$4u_2 = (u_3 + 20 + u_1 + 20)$$

$$u_1 - 4u_2 + u_3 = -40 \text{.................................(2)}$$

$$4u_3 = (u_4 + 20 + u_2 + 20)$$

$$u_2 - 4u_3 + u_4 = -40 \text{..........................(3)}$$

$$4u_4 = (100 + 20 + u_3 + 20)$$

$$u_3 - 4u_4 = -220 \text{..........................(4)}$$

Now (1)-4(2) gives $15u_2 - 4u_3 = 180$................(5)

4(3) +(4) gives $4u_2 - 15u_3 = -380 \ldots\ldots\ldots$.....(6)

Then 15(5) -4(6) gives $209u_2 = 4220$

$$u_2 = 20.2$$

From (5) we get

$$4u_3 = 15 \times 20.2 - 180$$

$$u_3 = 30.75$$

From(1)

$$4u_1 = 20 + 20.2$$

$$u_1 = 10.05$$

From (4)

$$4u_4 = 220 + 30.75$$

$$u_4 = 62.69$$

Thus The required values are $10.05, 20.2, 30.75, 62.69$

3. Use Crank Nicholson formula ,Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $given\ u(x,0) = 0, u(0,t) = 0, u(1,t) = t$

Solution :

By Using method , $k = ah^2 = 1\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Here a=1, $\therefore h = \frac{1}{4}$

We have

$u_{i,j+1} = \frac{1}{4}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j})$……………..(i)

	x→ *direction*					
		0	0.25	0.5	0.75	1
t	0	0	0	0	0	0
↓ *direct*	$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
	$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$
	$\frac{3}{16}$	0				$\frac{3}{16}$

$$u_1 = \frac{1}{4}(0+0+0+u_2) \qquad \therefore u_1 = \frac{u_2}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$$u_2 = \frac{1}{4}(0+0+u_1+u_3)$$

$$u_2 = \frac{1}{4}(u_1+u_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

$$u_3 = \frac{1}{4}\left(0+0+u_2+\frac{1}{16}\right)$$

$$u_3 = \frac{1}{4}\left(u_2+\frac{1}{16}\right) \ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Substitute u_1, u_3 values in (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}\left(2u_2+\frac{1}{16}\right)\right]$$

$$u_2 = \frac{1}{224} = 0.0045$$

$$u_2 = \frac{1}{896} = 0.0011$$

$u_3 = 0.0168 \qquad u_4 = 0.005899 \qquad u_5 = 0.0.01913$

$$u_6 = 0.0.05277$$