4.6. IIR FILTERS DESIGN-BILINEAR TRANSFORMATION

The bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. In this mapping all points in the left half of s plane are mapped inside the unit circle in the Z plane and all points in the right half of s-plane are mapped outside unit circle in the Z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration.

By the bilinear transformation

$$S\ Y(s) \rightarrow \frac{2\,(1-z^{-1})}{T\ (1+z^{-1})}\ \ Y(z)$$

Here in the s domain transfer function H(s) is transformed into Z domain transfer function by substituting s by the term $\frac{2\,(1-z^{-1})}{T\ (1+z^{-1})}$.

4.6.1. Relation between Analog and Digital poles in bilinear transformation:

The mapping of s domain function into z domain function by bilinear transformation is a one to one mapping, that is for every point in z plane ,there is exactly one corresponding point in s plane and vice versa.

The variable “s” represents a point on s plane and “z” is the corresponding point in z plane.

The following observations can be made

1. The point s_i lie on the left half of s plane the corresponding point in z plane will lie inside the unit circle in the z plane.
2. The point s_i lie on the right half of s plane the corresponding point in z plane will lie outside the unit circle in the z plane.
3. The point s_i lie on the imaginary axis of s plane the corresponding point in z plane will lie on the unit circle in the z plane.

The above discussions are applicable for mapping poles and zeros from s plane to z plane. The stability of the filter is associated with location of poles. We know that for a stable analog filter the poles should lie on the left half of s plane.

4.6.2. Frequency Warping

The change in properties when using Bilinear Transformation is referred to as Frequency Warping. The non-linear relationship between Ω and ω results in a distortion of the frequency axis. The spectral representation of frequency using Bilinear Transformation differs from the usual representation

4.6.3. Pre-Warping

Frequency warping follows a known pattern, and there is a known relationship between the warped frequency and the known frequency. We can use a technique called prewarping to account for the nonlinearity, and produce a more faithful mapping. You can remove the warping problem using a simple technique.

Example-1:

For the analog transfer function, H(s) = $\frac{2}{s^2+3s+2}$, determine H(z) using bilinear transformation if a) T= 1 second b) T = 0.1 second

Given that, H(s) = $\frac{2}{s^2+3s+2}$

Put, s = $\frac{2\,(1-z^{-1})}{T\;(1+z^{-1})}$

$$H(z) = \frac{2}{\left[\frac{2\,(1-z^{-1})}{T\,(1+z^{-1})}\right]^2 + 3\left[\frac{2\,(1-z^{-1})}{T\,(1+z^{-1})}\right] + 2}$$

$$= \frac{2T^2(1+z^{-1})^2}{4(1-z^{-1})^2 + 6T\,((1-z^{-2}) + 2T^2\,(1+z^{-1})^2}$$

When a) T=1 second

$$H(z) = \frac{2(1+z^{-1})^2}{4(1-z^{-1})^2 + 6(1-z^{-2}) + 2(1+z^{-1})^2}$$

$$= \frac{2(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 6(1-z^{-2}) + (1+2z^{-1}+z^{-2})2}$$

$$= \frac{2+4z^{-1}+2z^{-2}}{12-4z^{-1}}$$

$$= \frac{2+4z^{-1}+2z^{-2}}{12(1-\frac{4}{12}z^{-1})}$$

$$= \frac{0.1667+0.3333z^{-1}+0.1667z^{-2}}{1-0.333z^{-1}}$$

$$\mathbf{H(z)} = \frac{\mathbf{0.1667+0.3333z^{-1}+0.1667z^{-2}}}{\mathbf{1-0.333z^{-1}}}$$

When a) T=0.1 second

$$H(z) = \frac{2*0.1^2(1+z^{-1})^2}{4(1-z^{-1})^2+6*0.1((1-z^{-2})+2*0.1^2(1+z^{-1})^2}$$

$$=\frac{0.02(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2})+0.6(1-z^{-2})+0.02(1+2z^{-1}+z^{-2})}$$

$$=\frac{0.02z^{-1}+0.04z^{-1}+0.02z^{-1}}{4.62-7.96z^{-1}+3.42z^{-2}}$$

$$=\frac{0.0043+0.0087z^{-1}+0.0043z^{-2}}{1-1.7229z^{-1}+0.7403z^{-2}}$$

$$\mathbf{H(z)} = \frac{\mathbf{0.0043+0.0087z^{-1}+0.0043z^{-2}}}{\mathbf{1-1.7229z^{-1}+0.7403z^{-2}}}$$