5.3 SOLUTION OF STATE AND OUTPUT EQUATION IN CONTROLLABLE CANONICAL FORM

Consider the state equation of a linear time invariant system as,

$$\dot{X}(t) = AX(t) + BU(t)$$

The matrices A and B are constant matrices. This state equation can be of two types,

1. Homogeneous
2. Non-homogeneous

HOMOGENEOUS EQUATION

If A is a constant matrix and input control forces are zero then the equation takes the form

$$\dot{X}(t) = AX(t)$$

Such an equation is called homogeneous equation. The obvious equation is considered if input is zero. In such systems, the driving force is provided by the initial conditions of the system to produce the output. For example, consider a series RC circuit in which a capacitor is initially charged to V volts. The current is the output. Now there is no input control force, i.e., external voltage applied to the system. But the initial voltage on the capacitor drives the current through the system and capacitor starts discharging through the resistance, R. such a system works on the initial conditions without any input applied to it is called homogeneous system.

NON-HOMOGENEOUS EQUATION

If A is a constant matrix and matrix U(t) is non-zero vector i.e. the input control forces are applied to the system then the equation takes normal form as,

$$\dot{X}(t) = AX(t) + BU(t)$$

Such an equation is called non-homogeneous equation. Most of the practical systems require inputs to dive them. Such systems arc nonhomogeneous linear systems. The solution of the state equation is obtained by considering basic method of finding the solution of homogeneous equation.

STATE TRANSITION MATRIX

Properties of State Transition Matrix

1. $\phi(0) = e^{A\times 0} = I(unit\ matrix)$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

   or $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = \phi(t_1)\phi(t_2)$

Computation of State transition matrix

The state transition matrix, e^{At} can be computed by any one of the following two methods:

Method 1: Computation of e^{At} using matrix exponential

If the system matrix 'A' is an (n×n) square matrix, then each of these exponentials is an (n×n) square matrix of time functions, and one of the consequences of a theorem developed in linear algebra, known as the Cayley-Hamilton theorem, shows that such a matrix may be expressed as an $(n-1)^{st}$ degree polynomial in the matrix A.

That is,

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{i!}A^it^i$$

where, e^{At} – State transition matrix of order n x n

A – System matrix of order n x n

I – Unit matrix of order n x n

Method 2: Computation of e^{At} using Laplace transform

The theorem also states that the equation remains an equality if I is replaced by unity and A is replaced by any one of the scalar roots sI of the nth-degree scalar equation, det(sI-A) = 0. The expression det(sI-A) indicates the determinant of the matrix (sI-A). This determinant is an nth-degree polynomial ins. Let us assume that then roots are all different. This equation is called the characteristic equation of the matrix a, and the values of s which are the roots of the equation are known as the eigen values of A.

Consider the state equation without input vector,

$$\dot{X}(t) = AX(t)$$

On taking Laplace transform, we get,

$$sX(s) - X(0) = AX(s)$$

$$sX(s) - AX(s) = X(0)$$

$$sIX(s) - AX(s) = X(0)$$

$$(sI - A)X(s) = X(0)$$

Pre-multiplying both sides by $(sI-A)^{-1}$,

$$X(s) = (sI - A)^{-1}X(0)$$

On taking inverse Laplace transform,

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0)$$

On comparing with solution of state equation,

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

Also,

$$e^{At} = \phi(t)$$

where,

$$\phi(s) = (sI - A)^{-1}$$

which is the resolvent matrix.

Consider the state equation with input vector,

$$\dot{X}(t) = AX(t) + BU(t)$$

On taking Laplace transform, we get,

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$sIX(s) - AX(s) = X(0) + BU(s)$$

$$(sI - A)X(s) = X(0) + BU(s)$$

Pre-multiplying both sides by $(sI-A)^{-1}$,

$$X(s) = (sI - A)^{-1}X(0) + (sI - A)^{-1}BU(s)$$

$$X(s) = \phi(s)X(0) + \phi(s)BU(s)$$

On taking inverse Laplace transform,

$$x(t) = \phi(t)x(0) + L^{-1}[\phi(s)BU(s)]$$

Solution of output equation by Laplace Transform

$$Y(s) = CX(s) + DU(s)$$

$$y(t) = L^{-1}[CX(s) + DU(s)]$$

CONTROLLABLE CANONICAL FORM (CCF)

Probably the most straightforward method for converting from the transfer function of a system to a state space model is to generate a model in "controllable canonical form."

Consider a system defined by,

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

where u is the control input and y is the output. It can be written as,

$$\frac{Y(s)}{U(s)} = \frac{[b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n]}{[s^n + a_1 s^{n-1} + a_{n-1}s + a_n]}$$

Controllable canonical form of this system is given by,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \vdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad \cdots \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

OBSERVABLE CANONICAL FORM

The observable canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 0 & -a_n \\ 1 & 0 & 0 & \vdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \vdots & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & -a_2 \\ 0 & 0 & 0 & \vdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ b_{n-2} - a_{n-2} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

DIAGONAL CANONICAL FORM

There are cases where the dominator polynomial involves only distinct roots. For the distinct root case, we can write the equation in the form of

$$\frac{Y(s)}{U(s)} = \frac{[b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n]}{(s+p_1)(s+p_2)\ldots(s+p_n)} = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

The diagonal canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & 0 & 0 & \vdots & 0 \\ 0 & -p_2 & 0 & 0 & \vdots & 0 \\ 0 & 0 & -p_3 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad c_3 \quad \cdots \quad c_{n-1} \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$