Partial Differentiation

Let $u = f(x, y)$ be a function of two independent variables x and y, then

Differentiating 'u' with respect to 'x' keeping 'y' as a constant and it is denoted by $\frac{\partial u}{\partial x}$ or u_x ,

Similarly $\frac{\partial u}{\partial y}$ or u_y means differentiating 'u' with respect to 'y' keeping 'x' as a constant.

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are called first order partial derivatives.

Symbolically, if $u = u(x, y)$ then

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y}$$

Rule's of partial differentiation:

(i) Differential co-efficient of a sum:

If $u = v + w + \cdots$, where $v, w, \cdots$ are functions of $x, y, \cdots$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \cdots$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} + \cdots$$ and so on.

(ii) Differential co-efficient of a product:

If u and v are functions of x, y, z etc, then

$$\frac{\partial (uv)}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial (uv)}{\partial y} = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(iii) Differential co-efficient of a quotient:

If u and v are functions of x, y, z etc, then

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(iv)Derivative of a function:

If u is a function of t where t is a function of the variables $x, y, z \cdots$ then

$$\frac{\partial u}{\partial x} = \frac{du}{dt} \times \frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{du}{dt} \times \frac{\partial t}{\partial y}$$ and so on.

Successive partial Differentiation

Let $u = f(x,y)$ be a function of two independent variables x and y. Then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ will represent the first order partial derivative of 'u' with respect to 'x' and 'y'. Here both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are again in general a function of x and y. Hence each of these partial derivatives may again be differentiated with respect to 'x' and 'y' respectively and it is denoted by $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$

$u_{xx}= \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ = differentiating $\frac{\partial u}{\partial x}$ with respect to 'x' keeping 'y' as a constant.

$u_{yy}= \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ = differentiating $\frac{\partial u}{\partial y}$ with respect to 'y' keeping 'x' as a constant.

$u_{xy}= \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ = differentiating $\frac{\partial u}{\partial y}$ with respect to 'x' keeping 'y' as a constant.

Note:

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$ or $u_{xy} = u_{yx}$

Example:

If $\boldsymbol{u = (x-y)^4 + (y-z)^4 + (z-x)^4}$, show that $\boldsymbol{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$

Solution:

Given $u = (x-y)^4 + (y-z)^4 + (z-x)^4$

$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(z-x)^3(-1) \ldots (1)$$

$$\frac{\partial u}{\partial y} = 4(x-y)^3(-1) + 4(y-z)^3 \ldots (2)$$

$$\frac{\partial u}{\partial z} = 4(y-z)^3(-1) + 4(z-x)^3 \ldots (3)$$

$(1) + (2) + (3) \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$= 4(x-y)^3 - 4(z-x)^3 - 4(x-y)^3 + 4(y-z)^3 - 4(y-z)^3 + 4(z-x)^3 =$

Example:

If $\boldsymbol{f(x,y) = log\sqrt{x^2+y^2}}$, show that by $\boldsymbol{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$

Solution:

Given $f = log\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2x$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)1-x(2x)}{(x^2+y^2)^2}$$

$$= \frac{(y^2-x^2)}{(x^2+y^2)^2} \ldots(1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}} \times 2y$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)1-y(2y)}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)}{(x^2+y^2)^2} \ldots(2)$$

$$(1)+(2) \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{(y^2-x^2+x^2-y^2)}{(x^2+y^2)^2} = 0$$

Example:

If $r^2 = x^2 + y^2$ then $show$ that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ [AU May 2006]

Solution:

Given $r^2 = x^2 + y^2$

Differentiating partially with respect to 'x'

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1-x.\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r-x.\frac{x}{r}}{r^2}$$

$$= \frac{\frac{r^2-x^2}{r}}{r^2} = \frac{r^2-x^2}{r^3} \ldots(1)$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{r^2-y^2}{r^3} \ldots(2)$

$$(1)+(2) \Rightarrow \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$$

$$\Rightarrow \frac{r^2-x^2}{r^3} + \frac{r^2-y^2}{r^3} = \frac{r^2-x^2+r^2-y^2}{r^3}$$

$$= \frac{2r^2-(x^2+y^2)}{r^3}$$

$$= \frac{2r^2-r^2}{r^3}$$

$$= \frac{r^2}{r^3} = \frac{1}{r} = \text{L.H.S}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 = \left(\frac{x}{r}\right)^2 = \frac{x^2}{r^2}$$

Similarly $\left(\frac{\partial r}{\partial y}\right)^2 = \left(\frac{y}{r}\right)^2 = \frac{y^2}{r^2}$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (\because r^2 = x^2 + y^2)$$

$$\text{R.H.S} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

$$= \frac{1}{r} \times 1 = \frac{1}{r}$$

L.H.S = R.H.S