UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

PROBLEMS BASED ON DERIVATIVES FROM DIFFERENCE FORMULA

Numerical Differentiation and Integration

Numerical Differentiation	Numerical Integration
Newton's forward difference formula to compute derivatives (Equal interval)	Trapezoidal rule [$n = 1$ in Quadrature formula]
Newton's backward difference formula to compute derivatives(Equal interval)	Simpson's one third rule [$n = 2$ in Quadrature formula]
Lagrange's Interpolation formula (Equal or unequal intervals)	Simpson's three eighth rule [$n = 3$ in Quadrature formula]
Newton's divided difference Interpolation formula (Equal or unequal intervals)	Romberg's method
Maxima and minima of a tabulated function	Two point Gaussian's quadrature formula
	Three point Gaussian's quadrature formula
	Double integrals by Trapezoidal rule
	Double integrals by Simpson's 1/3 rule

Approximation of derivatives using interpolation polynomials

Numerical Differentiation

Given $y = y(x) = f(x)$ [in a table]

$\frac{dy}{dx} = y'(x) = f'(x)$ is the first numerical derivative

$\frac{d^2y}{dx^2} = y''(x) = f''(x)$ is the second numerical derivative

$\frac{d^ny}{dx^n} = y^{1\cdots n}(x) = f^{1\cdots n}(x)$ is the nth numerical derivative

Newton's forward difference formula to compute derivative

WKT, Newton's forward difference interpolation formula is

$$y(x) = f(x_0+uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \cdots$$

where $u = \frac{x-x_0}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{du}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\substack{\text{at } x = x_0 \\ \Rightarrow u = 0}} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\frac{du}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{du}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h}\frac{d}{du}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h^2}\frac{d^2y}{du^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{12u^2 - 36u + 22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\substack{\text{at } x = x_0 \\ \Rightarrow u = 0}} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{22}{24}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \left(\frac{24v+36}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\substack{\text{at } x = x_n \\ \Rightarrow v = 0}} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{36}{4!}\nabla^4 y_n + \cdots\right]$$

4.8.3 Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right] \quad (1)$$

substitute $h, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{ an equation in } u \quad (2)$$

Newton's Forward Formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \dots \dots \dots \dots ..$$

$u = \frac{x-x_0}{h}$ *and* $h = x_1 - x_0$

Newton's Backward Formula

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \dots \dots \dots \dots ..$$

$h = x_1 - x_0$ $v = \frac{x-x_n}{h}$

1. ***For the following data.***

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x=1.1

Given :

$x_0 = 1.0$ $\quad y_0 = 7.989$

$x_1 = 1.1$ $\quad y_1 = 8.403$

$x_2 = 1.2$ $\quad y_2 = 8.781$

x

$x_3 = 1.3$ $\quad y_3 = 9.129$

$x_4 = 1.4$ $\quad y_4 = 9.451$

$x_5 = 1.5$ $\quad y_5 = 9.750$

$x_6 = 1.6$ $\quad y_6 = 10.031$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
1.1	8.403	0.141 (Δy_0)	−0.036 ($\Delta^2 y_0$)				
				0.006 ($\Delta^3 y_0$)			
1.2	8.781	0.378	−0.030		-0.02		
		0.348				0.002	
1.3	9.129		-0.026	0.004	0.0		
		0.322					
1.4	9.451		-0.023	0.004		-0.001	-0.003
		0.299					
1.5	9.750		-0.018		-0.01		
		0.281		0.005			
1.6	10.031						

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta_{y_0} - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

$h = 0.1$

$$\left(\frac{dy}{dx}\right)_{x_0} = \left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[1.141 - \frac{1}{2}(-0.036) + \frac{1}{3}(0.006) - \frac{1}{4}(-0.02) + \frac{1}{5}(0.002) - \frac{1}{6}(-0.003)\right]$$

=3.946

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0\right]$$

$$= \frac{1}{(0.1)^2}\left[(-0.036) - (0.006) + \frac{11}{12}(-0.02) - \frac{5}{6}(0.002) + \frac{137}{180}(-0.003)\right]$$

= -3.545

The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds.

Time(sec)	0	5	10	15	20
Velocity (m/sec)	0	3	14	69	228

Find (a) Initial acceleration using the entire data (b) Final acceleration.

Solution: The difference table is

Time $t = x$	Velocity $v = y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
$0 (= x_0)$	$0 (= y_0)$				
		$3 (= \Delta y_0)$			
5	3		$8 \left(= \Delta^2 y_0\right)$		
		11		$36 \left(= \Delta^3 y_0\right)$	
10	14		44		$24 \left(= \Delta^4 y_0 \text{ or } \nabla^4 y_n\right)$
		55		$60 \left(= \nabla^3 y_n\right)$	
15	69		$104 \left(= \nabla^2 y_n\right)$		
		$159 (= \nabla y_n)$			
20	$228 (= y_n)$				

Here $x_0 = 0, h =$ interval length $= 5$

(a) WKT, acceleration $= \frac{dv}{dt} =$ rate of change of velocity

To find initial acceleration, put $\left(\frac{dv}{dt}\right)_{t=t_0} = \left(\frac{dv}{dt}\right)_{t=0}$.

i.e., initial acceleration exists at $t = 0 = x_0$ [which is nearer to beginning of the table], so we use

Newton's forward difference formula for first derivative.

$\therefore$ Newton's forward difference interpolation formula is

$$y(x) = f(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Now, Newton's forward difference formula for first derivative at $x = x_0 = 0$ [$\Rightarrow u = 0$]

i.e., $y'(x_0) = \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$ [Here $h = 5$]

$$= \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = 1$$

$\Rightarrow y'(0) = 1$

$\therefore$ Initial acceleration (acceleration when $t = 0$) is 1 m/sec^2.

Final acceleration exists at $t = 20 = x_4$ [Nearer to ending of table], so use Newton's backward difference interpolation formula for first derivative, put $\left(\frac{dv}{dt}\right)_{t=t_n} = \left(\frac{dv}{dt}\right)_{t=20}$.

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{5}\left[159 + \frac{1}{2}(104) + \frac{2}{6}(60) + \frac{6}{24}(24)\right]$$

$$= \frac{1}{5}(237) = 47.2 \text{ m / sec}^2$$

Find the value of $f'(8), f''(9)$, maximum and minimum value from the following data, using an approximate interpolation formula.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Solution: The values of x are unequally spaced.

To find $f(x)$, we use Newton's divided difference formula (or) Lagrange formula.

WKT, Newton's Divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \cdots \quad (1)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		
		310			
11	1210				

$$\therefore (1) \Rightarrow f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$$

$$= 48 + 52x - 208 + 15\left[x^2 - 9x + 20\right] + x^3 - x^2(16) + x(83) - 140$$

$$= x^3 + x^2(-16+15) + x(83 - 135 + 52) - 208 + 300 - 140 + 48$$

$$= x^3 - x^2 + x(0) + 0$$

$$\therefore f(x) = x^3 - x^2$$

$$f'(x) = 3x^2 - 2x, \quad f'(8) = 176$$

$$f''(x) = 6x - 2, \quad f''(9) = 52$$

<u>To find maximum and minimum</u>

$$\text{Put } f'(x) = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, x = \frac{2}{3}$$

At $x = 0,\ f''(x=0) = 6(0) - 2 = -2 < 0$

$\Rightarrow x = 0$ is a maximum point & maximum value is $f(x=0) = 0$.

At $x = \frac{2}{3},\ f''\left(x = \frac{2}{3}\right) = 6\left(\frac{2}{3}\right) - 2 = 2 > 0.$

$\Rightarrow x = \frac{2}{3}$ is a minimum point & minimum value is $f\left(x = \frac{2}{3}\right) = -\frac{4}{27}$

Evaluate y' and y'' at $x = 2$ given

x	0	1	3	6
y	18	10	−18	40

Solution:

$$\left[\text{Ans}: y(x) = x^3 - \frac{70}{9}x^2 - \frac{15}{9}x + 18, y'(x=2) = -\frac{187}{9}, y''(x=2) = -\frac{22}{9}\right]$$

Find the value of $\cos(1.747)$ using the values given in the table below :

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9916	0.9857	0.9781	0.9691	0.9584

Solution: [Ans : -0.175]

Find $\sec 31°$ from the following data :

θ	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

Solution: $\left[\text{Ans : } \sec^2 31 = 1.3835 \Rightarrow \sec 31 = 1.174, \text{ Hint : } 1° = \frac{\pi}{180} = 0.017453292\right]$

$2.$ ***Find the value of y when x = 2 for the following data.***

x	0	5	10	15
y	14	379	1444	3584

Given :

$x_0 = 0 \qquad y_0 = 14$

$x_1 = 5 \qquad y_1 = 379$

$x_2 = 10 \qquad y_2 = 1444$

$x_3 = 15 \qquad y_3 = 3584$

$$h = x_1 - x_0 = 5 - 0 = 5$$

$$u = \frac{x - x_0}{h} = \frac{x - 0}{5} = \frac{2 - 0}{5} = \frac{2}{5} = 0.4$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	14			
		$379-14=365$ (Δy_0)		
5	379		$1065-365=700$ $(\Delta^2 y_0)$	
				$1075-700$ $=375$ $(\Delta^3 y_0)$
10	1444	$1444-379=1065$		
			$2140-1065=1075$	
15	3584			
		$3584-1444=2140$		

$$y(x)=y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$y(2)=14+\frac{0.4}{1!}(365)+\frac{0.4(0.4-1)}{2!}(700)+\frac{0.4(0.4-1)(0.4-2)}{3!}(375)$$

$$y(2)=14+146+\frac{(-0.24)}{2}(700)+\frac{(-0.384)}{6}(375)$$

$$y(2)=14+146-84+24=100$$

<u>Newton's Backward Formula</u>

$$y(x)=y_n+\frac{v}{1!}\nabla y_n+\frac{v(v+1)}{2!}\nabla^2 y_n+\frac{v(v+1)(v+2)}{3!}\nabla^3 y_n+\cdots\ldots\ldots\ldots\ldots..$$

$h=x_1-x_0 \quad v=\frac{x-x_n}{h}$

3. ***Find the value of*** y ***when and*** $x = 43$ & $x = 84$ ***for the following data.***

x	40	50	60	70	80	90
y	184	204	226	250	276	304

Solution:

$x_0 = 40$ $y_0 = 184$

$x_1 = 50$ $y_1 = 204$

$x_2 = 60$ $y_2 = 226$

$x_3 = 70$ $y_3 = 250$

$x_4 = 80$ $y_4 = 276$

$x_5 = 90$ $y_5 = 304$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	184					
		$204 - 184 = 20$				
50	204					
			2			
		$226 - 204 = 22$				
60				0		
	226					
			2			
70					0	
				0		0
		$250 - 226 = 24$				
	250					
					0	
			2			
80						
				0		
	276					
		$276 - 250 = 26$				
90						
			2			
		$304 - 276 = 28$				
	304					

(i) when x = 43

$$h = x_1 - x_0 = 5 - 40 = 10$$

$$u = \frac{x - x_0}{h} = \frac{x - 40}{10} = \frac{43 - 40}{10} = \frac{3}{10} = 0.3$$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$y(43) = 184 + \frac{0.3}{1!}(20) + \frac{0.3(0.3-1)}{2!}(2) + \frac{0.3(0.3-1)(0.3-2)}{3!}(0)$$

$$y(43) = 184 + 6 + \frac{0.3(-0.7)}{2}(2) + 0$$

$$y(43) = 184 + 6 - 0.21$$

$$y(43) = 189.79$$

(ii) when x = 84

$$h = x_1 - x_0 = 50 - 40 = 10$$

$$v = \frac{x - x_n}{h} = \frac{x - 90}{10} v = \frac{84 - 90}{10} = \frac{-6}{10} = -0.6$$

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \ldots\ldots\ldots\ldots$$

$$y(84) = 304 + \frac{(-0.6)}{1!}(28) + \frac{(-0.6)(-0.6+1)}{2!}(2) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0) + \cdots$$

$$y(84) = 304 - 16.8 + \frac{(-0.6)(0.4)}{2}(2) + 0$$

$$y(84) = 304 - 16.8 - 0.24$$

$y(84) = 286.96$

Anna University Questions

1. Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(ND10)

Solution : [Ans: $\tan 45°15' = 1.00876$, by Newton's forward difference formula]

2. Derive Newton's backward difference formula by using operator method. (MJ12)

3. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table:

x :	4	6	8	10
y :	1	3	8	16

(ND12)

Solution : [$y(x = 5) = 1.625$, by Newton's forward difference formula]

4. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below. (8)

x :	0	1	2	3
y :	1	2	1	10

(MJ13)

Solution :

$y(x) = 2x^3 - 7x^2 + 6x + 1$, by Newton's forward & backward formula

$y(x = 4) = 41$