5.3 CONVERSION OF TRANSFER FUNCTIONS TO STATE VARIABLE MODELS

In canonical form (or normal form) of state model the system matrix A will be diagonal matrix. The elements on the diagonal are the poles of the transfer function of the system.

By partial fraction expansion the transfer function Y(s) /U(s) of the nth order system can be expressed as shown in equation

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s+\lambda_1} + \frac{C_2}{s+\lambda_2} + \cdots + \frac{C_n}{s+\lambda_n}$$

Here C_1, C_2, C_3,……C_n are residues and $\lambda_1, \lambda_2, \lambda_3 \ldots . \lambda_n$ are roots of denominator polynomial.

The above equation can be rearranged as

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s(1+\frac{\lambda_1}{s})} + \frac{C_2}{s(1+\frac{\lambda_2}{s})} + \cdots + \frac{C_n}{s(1+\frac{\lambda_n}{s})}$$

$$Y(s) = b_0\, U(s) + \left[\frac{1/s}{1+(1/s)\lambda_1} X\, C_1\right] U(s)$$

$$+ \left[\frac{1/s}{1+(1/s)\lambda_2} X\, C_2\right] U(s) \ldots\ldots\ldots . \left[\frac{1/s}{1+(1/s)\lambda_n} X\, C_n\right] U(s)$$

The equation can be represented by a block diagram as shown in fig 5.1

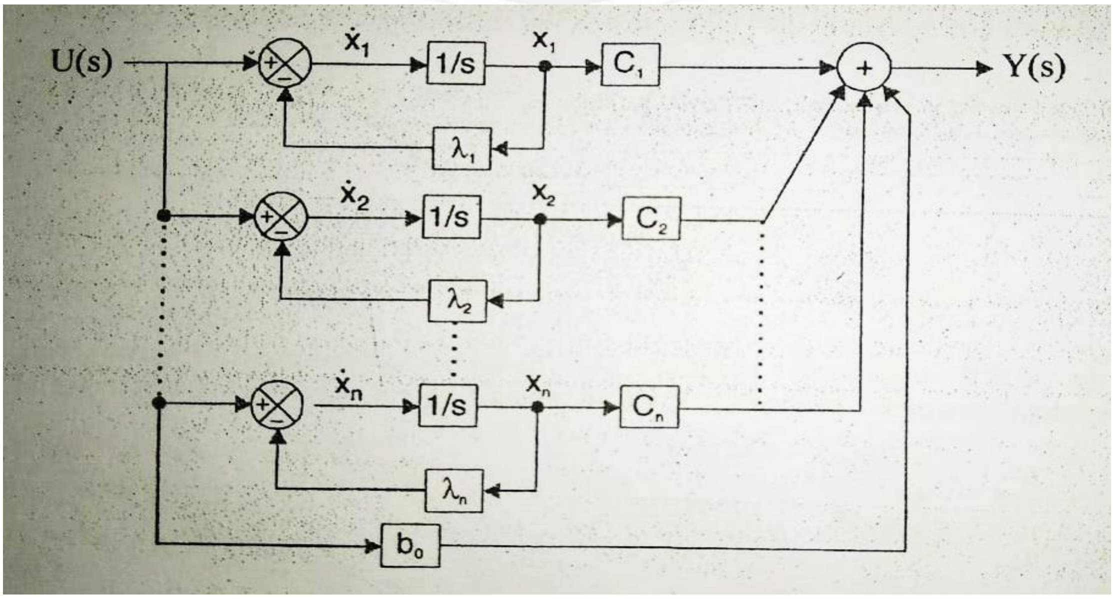

Figure 5.1: ***Block diagram of canonical state model***

[Source: "Control System Engineering" by Nagoor Kani, page-6.31]

Assign state variables at the output of integrator. The input of the integrator will be first derivative of state variable. the state equations are formed by adding the incoming signals to the integrator and equating to first derivative of state variable.

The state equations are,

$$\dot{x_1} = -\lambda_1 x_1 + u$$

$$\dot{x_2} = -\lambda_2 x_2 + u$$

$$\dots$$

$$\dot{x_n} = -\lambda_n x_n + u$$

The output equation is,

$$y = C_1 x_1 + C_2 x_2 + \cdots . C_n x_n + C_1 u$$

The canonical form of state model in the matrix form is given below

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \dot{x_n} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [u]$$

$$y = [C_1 \quad C_2 \quad C_3 \quad \dots \quad C_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \dot{x_n} \end{bmatrix} + [b_0][u]$$

The advantage of canonical form is that the state equation are independent of each other. The disadvantage is that the canonical variables are not physical variables and so they are not available for measurement and control.

Consider a system with poles $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_n$, where $\lambda_1$has multiplicity of three. The input matrix (B) and system matrix for this case will be in equation. the system matrix also denoted as J

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}; A = J = \begin{bmatrix} -\lambda_1 & 1 & 0 & & \dots & 0 \\ 0 & -\lambda_1 & 1 & & \dots & 0 \\ 0 & 0 & -\lambda_1 & & \dots & 0 \\ 0 & 0 & 0 & -\lambda_4 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & & -\lambda_n \end{bmatrix}$$

The transfer function of the system for this case is given by above equation and the block diagram is shown in figure 5.3.1

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+\lambda_1)^3} + \frac{C_2}{(s+\lambda_1)^2} + \frac{C_3}{s+\lambda_1} + \frac{C_4}{s+\lambda_4} \dots + \frac{C_n}{s+\lambda_n}$$

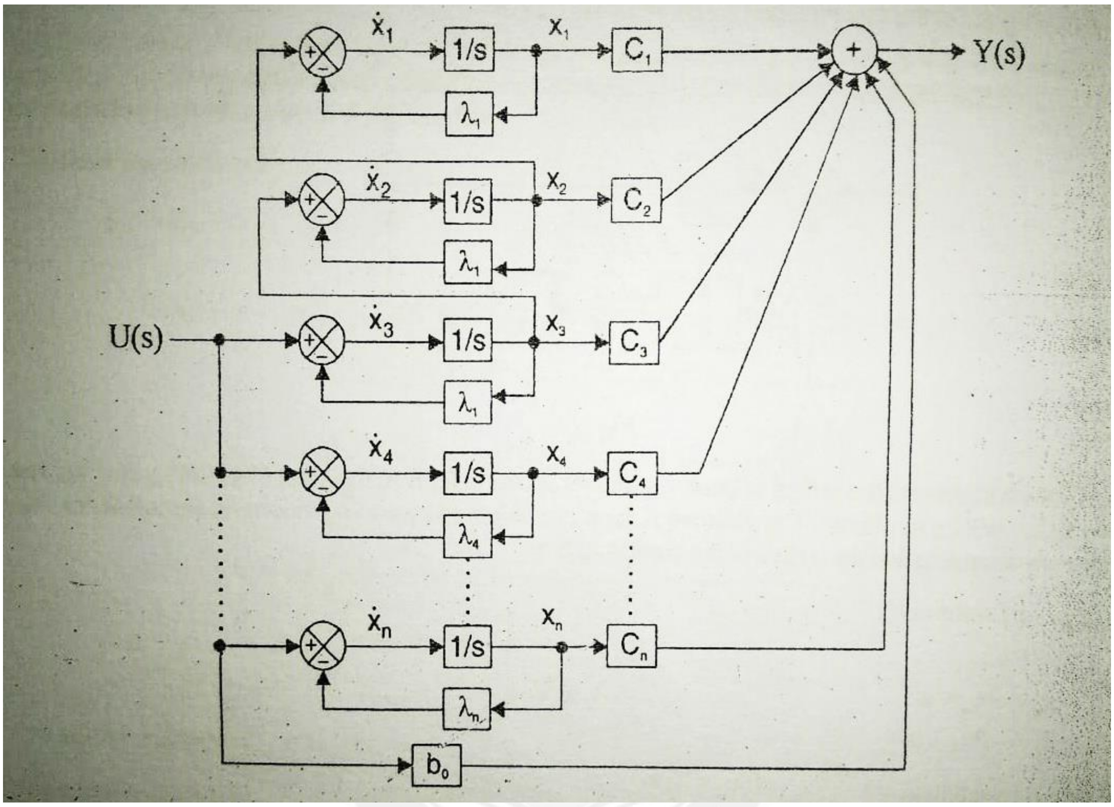

Figure 5.3.1: Block diagram of Jordan canonical state model

[Source: "Control System Engineering" by Nagoor Kani, page-6.32]

The state model of a system is not unique and it can be formed using physical variables, phase variables or canonical variables. When anon-diagonal system matrix A has distinct Eigen values, it an be converted to diagonal matrix by a similarity transformation using model matrix, M.

CANONIAL FORM OF STATE MODEL

Consider the state equation of a system$\dot{X}(t) = A\,X(t) + B\,U(t)$, Where X (t) = state vector of order (n x 1).let us define a new state variable vector Z, such that X=MZ, where M is the modal matrix or diagonalization matrix.

The state model of nth order system is given by,

$$\dot{X} = A\,X + B\,U$$

$$Y = C\,X + D\,U$$

On substituting X = MZ in the state model of the system, we get

$$\dot{X} = A\,MZ + B\,U$$

$$Y = C\,MZ + D\,U$$

Premultiply equation by M^{-1}

$$M^{-1}\dot{X} = M^{-1}A\,MZ + M^{-1}B\,U - - - - (1)$$

The relation governing X and Z is, X = MZ

On differentiating above equation we get $\dot{X} = M\dot{Z}$

premultiplying the above equation by M^{-1}

$$M^{-1}\dot{X} = \dot{Z} - - - - - - - (2)$$

From equation (1) and (2), we get,

$$\dot{Z} = M^{-1}AMZ + M^{-1}BU - - - - (3)$$

Let $M^{-1}AM = A$

$$M^{-1}B = \tilde{B}$$

$$CM = \tilde{C}$$

The transformed state model is given by

$$\dot{Z} = AZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

The model matrix M is obtained from Eigen vectors. When the system matrix A is in the companion or bush form then the model matrix is given by a special matrix called Vander monde matrix, V

$$A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix}$$

$$then \;\; M = V = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 & \dots & {\lambda_n}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {\lambda_1}^{n-1} & {\lambda_2}^{n-1} & \dots & \dots & {\lambda_n}^{n-1} \end{bmatrix}$$

JORDAN CANONICAL FORM OF STATE MODEL

If the Eigen value has multiplicity then the system matrix cannot be diagonalized. In this case the transformation as Jordan matrix J, where $J=M^{-1}AM$.

$$\dot{Z} = JZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

The Jordan matrix J will have a Jordan block of size q x q correspond to a Eigen value of λ_1 with multiplicity q.

$$J = \begin{bmatrix} -\lambda_1 & 1 & 0 & \dots & 0 \\ 0 & -\lambda_1 & 1 & \dots & 0 \\ 0 & 0 & -\lambda_1 & \dots & 0 \\ 0 & 0 & 0 & -\lambda_4 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_n \end{bmatrix}$$

$$M = V = \begin{bmatrix} 1 & 0 & 0 & \dots & 1 \\ \lambda_1 & 1 & 0 & \dots & \lambda_n \\ {\lambda_1}^2 & 2\lambda_1 & 1 & \dots & {\lambda_n}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {\lambda_1}^{n-1} & n{\lambda_1}^{n-1} & \dots & \dots & {\lambda_n}^{n-1} \end{bmatrix}$$

COMPUTATION OF STATE TRANSITION MATRIX BY CANONICAL TRANSFORMATION

Consider the state equation without input,

$$\dot{X}(t) = A\,X(t)$$

The solution of state equation is

$$X(t) = e^{At}X(0)$$

The matrix e^{At}can be expressed as infinite series

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$