UNITV

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

PROBLEMS BASED ON TWO DIMENSIONAL POISSON'S EQUATIONS

Solve the POISSON'S equation $u_{xx} + u_{yy} = -81xy, 0 < x < 1, 0 < y < 1$ **Given that**

$u(0, y) = 0$; $u(1, y) = 100$,

$u(x, 0) = 0$, $u(x, 1) = 100$ and $h = \frac{1}{3}$

Solution :

Here $h = \frac{1}{3}$

The standard five point formula for the given equation is

$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_i, j = h^2 f(ih, jh) = h^2[-81(ih, jh)] = h^4(-81)ij = -ij$……………………………………(i)

For $u_1(i = 1, j = 2)$,.(i) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

$-4u_1 + u_2 + u_3 = -102$……………………………………(ii)

For $u_2(i = 2, j = 2)$,.(i) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

$u_1 - 4u_2 + u_4 = -204$ ………………………………………………(iii)

For $u_3(i = 1, j = 1)$,.(i) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

$u_1 - 4u_3 + u_4 = -1$…………………………………………………(iv)

For $u_4(i = 2, j = 1)$,.(i) gives $u_3 + 100 + u_2 - 4u_4 = -2$

$u_2 + u_3 - 4u_4 = -102$…………………………………………………(v)

Y

0	100 U1	100 U2	100
0	U3	U4	100
0	0	0	100

X

Subtracting from (v) from (ii) $-4u_1 + 4u_4 = 0$

$\therefore u_1 = u_4$

Then (iii) becomes $2u_1 - 4u_2 = -204$ and (iv) becomes $2u_1 - 4u_3 = -1$

Now (4)$\times$ $(ii) + (vi)\ gives - 14u_1 + 4u_3 = -612$

$$(vii) + (viii)\ gives - 12u_1 = -613$$

Thus $\quad u_1 = \frac{613}{12} = 51.0833 = u_4$

From (vi), $\quad u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$

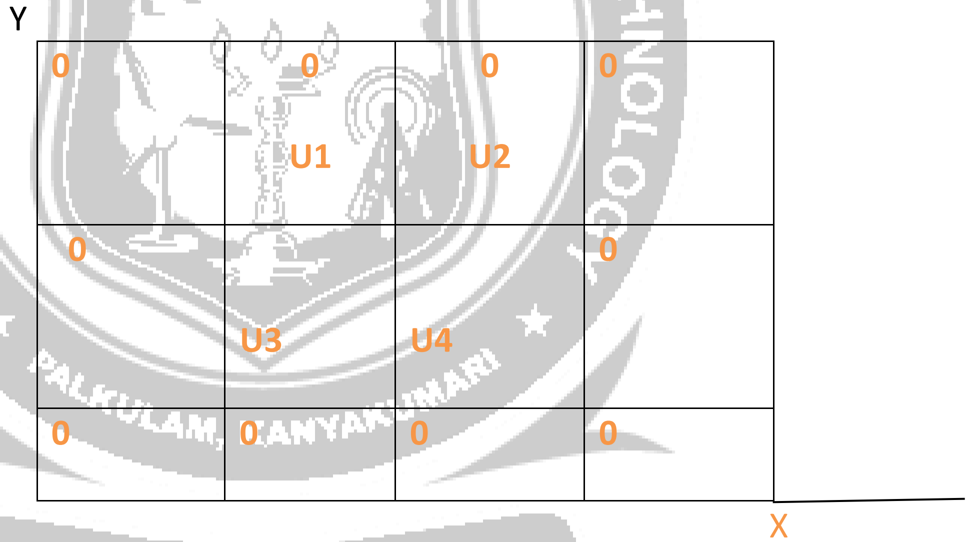
From (vii), $\quad u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$

2.Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ $with$ $u = 0$ on the boundary and mesh length h=1

Solution : Here h=1

The standard five point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \dots\dots\dots\dots\dots (i)$$



For $u_1 (i = 1, j = 2)$,.(i) gives $0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

For $u_2 (i = 2, j = 2)$,.(i) gives $u_2 = \frac{1}{4}(u_1 + u_4 + 180) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iii)$

For $u_3 (i = 1, j = 1)$,.(i) gives $u_3 = \frac{1}{4}(u_1 + u_4 + 120) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$ (iv)

For $u_4 (i = 2, j = 1)$,.(i) gives

$$u_4 = \frac{1}{4}(u_2 + u_3 + 150) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (ii)$$

Equations (ii) and (iii) show that $u_4 = u_1$ Thus the above equations reduces to

$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

$$u_2 = \frac{1}{4}(u_2 + 90)$$

$$u_3 = \frac{1}{4}(u_1 + 60)$$

Now let us solve these equations by Gauss Seidal method

First iteration : Starting from the approximations $u_2 = 0, u_3 = 0$ we obtain

$$\boldsymbol{u_1^{(1)} = 37.5}$$

Then$\boldsymbol{u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64}$

$$\boldsymbol{u_3^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49}$$

Second iteration :

$$\boldsymbol{u_1^{(2)} = \frac{1}{2}(64 + 49 + 150) = 66}$$

Then$\boldsymbol{u_2^{(2)} = \frac{1}{2}(66 + 90) \approx 78}$

$$\boldsymbol{u_3^{(2)} = \frac{1}{2}(66 + 60) \approx 63}$$

Third iteration :

$$\boldsymbol{u_1^{(3)} = \frac{1}{2}(78 + 63 + 150) \approx 73}$$

Then$\boldsymbol{u_2^{(3)} = \frac{1}{2}(73 + 90) \approx 78}$

$$\boldsymbol{u_3^{(3)} = \frac{1}{2}(73 + 60) \approx 67}$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{2}(82 + 67 + 150) = 75$$

Then $u_2^{(4)} = \frac{1}{2}(75 + 90) \approx 82.5$

$$u_3^{(4)} = \frac{1}{2}(75 + 60) \approx 67.5$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{2}(82.5 + 67.5 + 150) = 75$$

Then $u_2^{(5)} = \frac{1}{2}(75 + 90) \approx 82.5$

$$u_3^{(5)} = \frac{1}{2}(75 + 60) \approx 67.5$$

$u_1 = 75$

$u_2 = 82.5$

$u_3 = 67.5$

$u_4 = 75$