5.4 PROBLEMS ON AUTOCORRELATION FUNCTION OF INPUT AND OUTPUT

1. **A random process $\{X(t)\}$ is applied to a network with response $h(t) = te^{-bt}u(t)$, where $b > 0$ is a constant. The cross function of $X(t)$ with the output $Y(t)$ is known to have the same $i.e$ ACF $\cdot$ $R_{XY}(\tau) = \tau e^{-b\tau}u(\tau)$. Find the ACF of the output $\{Y(t)\}$.**

Sol: i) To find (ω) :

$$h(t) = te^{-bt}u(t)H(\omega) = F[h(t)]$$

$$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} te^{-bt}u(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{m} te^{-bt}(1)e^{-i\omega t}dt = \int_{0}^{\infty} te^{-(b+\mathrm{i}\omega)\mathrm{t}}dt$$

$$H(\omega) = \left[t\frac{e^{-(b+\mathrm{i}\omega)t}}{-(b+i\omega)} - (1)\frac{e^{-(b+\mathrm{i}\omega)t}}{(b+i\omega)^2}\right]_0^{\infty} = \frac{1}{(b+i\omega)^2}$$

$$\text{Also } H^*(\omega) = \frac{1}{(b-i\omega)^2}$$

ii) To find $S_{XY}(\omega)$:

The cross power spectrum of $\{X(t)\}$ and $[Y(t)]$ is given by

$$
\begin{aligned}
S_{XY}(\omega) &= F[R_{XY}(t)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-i\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} \tau e^{-b\tau}u(\tau)e^{-i\omega\tau}d\tau = \left[\int_{-\infty}^{0} 0 + \int_{0}^{\infty} \tau e^{-b\tau}(1)e^{-i\omega\tau}d\tau\right] \\
&= \int_{0}^{\infty} \tau e^{-(b+i\omega)\tau}d\tau = \left[\tau\frac{e^{-(b+i\omega)t}}{-(b+i\omega)}\right] - \left[\frac{e^{-(b+i\omega)\tau}}{(-(b+i\omega))^2}\right]_0^{\infty} \\
S_{XY}(\omega) &= \frac{1}{(b+i\omega)^2}
\end{aligned}
$$

iii) To find $R_{YY}(\tau)$:

We know that,

$$
\begin{aligned}
S_{YY}(\omega) &= H^*(\omega)S_{XY}(\omega) \\
&= \frac{1}{(b-i\omega)^2}\frac{1}{(b+i\omega)^2} = \frac{1}{((b-i\omega)(b+i\omega))^2} \\
&= \frac{1}{(b^2+\omega^2)^2}
\end{aligned}
$$

The ACF of the output $Y(t)$ is given by

$$
\begin{aligned}
R_{YY}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(b^2+\omega^2)^2}e^{i\omega\tau}d\tau
\end{aligned}
$$

$$
= \frac{1}{2\pi}\frac{\pi}{2b^3}[1+b|\tau|]e^{-b|\tau|} \; | \because \int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+a^2)^2}d\omega = \frac{\pi}{2a^3}(1+a|\tau|)e^{-a|\tau|}
$$

$$
R_{YY}(\tau) = \frac{1}{4b^3}[1+b|\tau|]e^{-b|\tau|}
$$

2. **Assume a random process $\{X(t)\}$ is given to a system with for system transfer function $H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_0 \\ 0 & \text{else} \end{cases}$**

If the ACF of input is $\frac{N_0}{2}\delta(\tau)$, find the ACF of output.

Sol: i) To find $S_{XX}(\omega)$:

Given the ACF of the input is

$$R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$$

The PSD of the input is given by

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-i\omega\tau}d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau \\ &= \frac{N_0}{2}(1) \end{aligned}$$

$$\text{Given } H(\omega) = \begin{cases} 1 & ; |\omega| \leq \omega_0 \\ 0 & ;\text{else} \end{cases}$$

ii) To find $R_{YY}(\tau)$: The relation between the PSD's of input and output is given by

$$\begin{aligned} S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\ &= (1)^2\frac{N_0}{2};\ |\omega| \leq \omega_0 \\ \therefore S_{YY}(\omega) &= \frac{N_0}{2}; |\omega| \leq \omega_0 \end{aligned}$$

The ACF of the output is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}\frac{N_0}{2}e^{i\omega\tau}d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau}d\omega \qquad = \frac{N_0}{4\pi}\left[\frac{e^{i\omega r}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi i\tau}\left[e^{t\omega_0 t} - e^{-\omega\omega_0 \mathrm{r}}\right] \qquad = \frac{N_0}{4\pi i\tau}2i\sin\omega_0\tau$$

$$R_{YY}(\tau) = \frac{N_0}{2\pi\tau}\sin\omega_0\tau$$

3. An LTI system has an impulse response $h(t) = e^{-\beta t}u(t)$. Find the output auto correlation $R_{YY}(\tau)$ corresponding to an input $X(t)$

Sol: i) To find (ω) :

Given $h(t) = e^{-\beta t}u(t)$

$$H(\omega) = F[h(t)]$$

$$= \int_{-\infty}^{\infty} e^{-\beta t}u(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} e^{-\beta t}(1)e^{-iat}dt = \int_{-\infty}^{\infty} e^{-(\beta+i\omega)t}dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)}\right]_0^{\infty}$$

$$|H(\omega)| = \frac{1}{\sqrt{\beta^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}$$

ii) To find $R_{YY}(\tau)$:

The relation between the PSD's of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{1}{\beta^2+\omega^2} S_{XX}(\omega)$$

$$R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)] \ [\therefore S_{XX}(\omega) \text{ not known}]$$

$$= F^{-1}\left[\frac{1}{\beta^2+\omega^2}\right] \cdot F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\beta^2+\omega^2} e^{-\omega\tau}dt * R_{XX}(\tau)$$

$$= \frac{1}{2\pi}\frac{\pi}{\beta}e^{-\beta|\tau|} * R_{XX}(\tau) = \frac{1}{2\beta}e^{-\beta|\tau|} * R_{XX}(\tau)$$

$R_{YY}(\tau) = \int_{-\infty}^{\infty} \frac{1}{2\beta} e^{-\beta|u|} R_{XX}(\tau-u)du$ (Since $f(t) * g(t) = \int_0^t f(u)g(t-$

$u)du$)

4. The relation between input{X(t) }and output {Y(t)} of the diode is expressed as $Y(t) = X^2(t)$. Let {X(t)} be a zero mean stationary Gaussian random process with ACF $R_{XX}(\tau) = e^{-\alpha|\tau|}; \alpha > 0$. Find the output auto correlation $R_{YY}(\tau)$ of the input.

Sol: Given $Y(t) = X^2(t)$, where X(t) is the zero mean stationary Gaussian random process.

$$R_{YY}(\tau) = E[Y(t_1)\ Y(t_2)] = E[X^2(t_1)\ X^2(t_2)]$$

$$= E[(X(t_1)X(t_2))^2]$$

$$= E[X^2(t_1)]\ E[X^2(t_2)] + 2\ [E[(X(t_1)X(t_2)]]^2$$

$$= R_{XX}(0)\ R_{XX}(0) + 2\ [R_{XX}(\tau)\ ]^2 \qquad \ldots\ldots\ldots\ldots(1)$$

Given $R_{XX}(\tau) = e^{-\alpha|\tau|}$

$$R_{XX}(0) = e^0 = 1$$

$$(1) \Rightarrow R_{YY}(\tau) = 1+2\left[e^{-\alpha|\tau|}\right]^2$$

$$= 1+2\ e^{-\alpha|\tau|}$$

The PSD of output is given by

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)\ e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 1 + 2\ e^{-2\alpha|\tau|}\ e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau + 2\int_{-\infty}^{\infty} 1 + 2\ e^{-2\alpha|\tau|}\ d\tau$$

$$= 2\pi\delta(\omega) + 2\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\ [\cos\omega\tau - i\sin\omega\tau] d\tau$$

$$=2\pi\delta(\omega)+2\left[\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\sin\omega\tau\, d\tau\right]$$

$$=2\pi\delta(\omega)+2\left[2\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\, d\tau - i(0)\right]$$

$$=2\pi\delta(\omega)+4\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\tau\, d\tau$$

$$S_{YY}(\omega) = 2\pi\delta(\omega)+\frac{8\alpha}{4\alpha^2+\omega^2} \qquad \int_0^{\infty} e^{-at}\cos bt\, dt = \frac{a}{a^2+b^2}$$

5.Find the auto correlation of the band limited white noise {N(t)} with PSD

given by $S_{NN}(\omega) = \frac{N_0}{2}; |\omega-\omega_0| < \omega_B;$

Sol: $S_{NN}(\omega)=\frac{N_0}{2}; |\omega-\omega_0| < \omega_B;$

$|\omega-\omega_0| < \omega_B \Rightarrow -\omega_B < \omega-\omega_0 < \omega_B < \omega < \omega_0+\omega_B$

The ACF of {N(t)} is given by

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\, e^{i\omega\tau} d\omega$$

$$=\frac{1}{2\pi}\int_{\omega_0-\omega_B}^{\omega_0+\omega_B}\frac{N_0}{2}e^{i\omega\tau}d\omega$$

$$=\frac{N_0}{4\pi}\int_{\omega_0-\omega_B}^{\omega_0+\omega_B} e^{i\omega\tau}d\omega$$

$$=\frac{N_0}{4\pi}\left(\frac{e^{i\omega\tau}}{i\tau}\right)_{\omega_0-\omega_B}^{\omega_0+\omega_B}$$

$$=\frac{N_0}{4\pi}\frac{\left(e^{i(\omega_0+\omega_B)\tau}-e^{i(\omega_0-\omega_B)\tau}\right)}{i\tau}$$

$$=\frac{N_0}{4\pi}\frac{e^{i\omega_B\tau+i\omega_B\tau}-e^{i\omega_B\tau-i\omega_B\tau}}{i\tau}$$

$$=\frac{N_0}{4\pi}\frac{e^{i\omega_0\tau}e^{i\omega_B\tau}-e^{i\omega_0\tau}e^{i\omega_B\tau}}{i\tau}$$

$$=\frac{N_0}{4\pi} e^{i\omega_0\tau} \frac{e^{i\omega_B\tau}-e^{-i\omega_B\tau}}{i\tau}$$

$$=\frac{N_0 e^{i\omega_0\tau}}{4\pi} \frac{2i\, sin\omega_B\tau}{i\tau}$$

$$=\frac{N_0}{4\tau\pi} e^{i\omega_0\tau} sin\omega_B\tau$$

$$=\frac{N_0}{4\tau\pi} (cos\omega_0\, \tau + i\, sin\omega_0\, \tau) sin\omega_B\, \tau$$

$$=\frac{N_0}{2\tau\pi}\; cos\omega_0\, \tau\; sin\omega_B\, \tau + i \frac{N_0}{2\tau\pi}\; sin\omega_0\, \tau\; sin\omega_B\, \tau$$

Since ACF is real, equating the real part, we get

$$R(\tau) = \frac{N_0}{2\tau\pi}\; cos\omega_0\, \tau\; sin\omega_B\, \tau$$

$$=\frac{N_0}{2\pi}\; cos\omega_0\, \tau\; \frac{sin\omega_B\, \tau}{\omega_B\, \tau} \omega_B$$

$$R(\tau) = \frac{N_0\omega_B}{2\tau\pi}\; cos\omega_0\, \tau\; \frac{sin\omega_B\, \tau}{\omega_B\, \tau}$$