UNIT – II ONE DIMENSIONAL PROBLEMS

PART - A

1. **What is truss?*(May/June 2014)***
   A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements. Truss can able to transmit load and it can deform only along its length. Loads are acting only at the joints.

2. **State the assumptions made in the case of truss element.**
   The following assumptions are made in the case of truss element,
   1. All the members are pin jointed.
   2. The truss is loaded only at the joints
   3. The self weight of the members are neglected unless stated.
3. **What is natural co-ordinate?*(Nov/Dec 2014), (April/May 2011)***
   A natural co-ordinate system is used to define any point inside the element by a set of dimensionless numbers, whose magnitude never exceeds unity, This system is useful inassembling of stiffness matrices.
4. **Define shape function. State its characteristics *(May/June 2014), (Nov/Dec 2014), (Nov/Dec 2012)***
   In finite element method, field variables within an element are generally expressed by the following approximate relation:
   u (x,y) = N1(x,y) u1+N2 (x,y) u2+ N3(x,y) u3
   Where u,1 u2, u3 are the values of the field variable at the nodes and N1 N2 N3 are interpolation function. N1 N2 N3 is called shape functions because they are used to express the geometry or shape of the element.
   The characteristics of the shape functions are follows:
   1. The shape function has unit value at one nodal point and zero value at the other nodes.
   2. The sum of the shape function is equal to one.
5. **Why polynomials are generally used as shape function?**
   Polynomials are generally used as shape functions due to the following reasons:
   1. Differentiation and integration of polynomials are quite easy.
   2. The accuracy of the results can be improved by increasing the order of the Polynomial.
   3. It is easy to formulate and computerize the finite element equations.
6. **Write the governing equation for 1D Transverse and longitudinal vibration of the bar at one end and give the boundary conditions. *(April/May 2015)***
   The governing equation for free vibration of abeam is given by,

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0$$

   Where,
   E – Young's modulus of the material.
   I – Moment of inertia
   P – Density of the material.
   A – Cross sectional area of the section of beam.

   The governing equation for 1D longitudinal vibration of the bar at one end is given by

$$\frac{d^2U}{dx^2}\ AE + \rho AU\omega^2 = 0$$

   Where,
   U – axial deformation of the bar (m)
   ρ – Density of the material of the bar (kg/m^3)

ω – Natural frequency of vibration of the bar

A – Area of cross section of the bar (m^2)

7. Express the convections matrix for 1D bar element. *(April/May 2015)*

$$\frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Convection stiffness matrix for 1D bar element:

$$\frac{hPT_aL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Convection force matrix for 1D bar element:

Where,

h- Convection heat transfer coefficient (w/m2k)

P – Perimeter of the element (m)

L – Length of the element (m)

Ta – Ambient temperature (k)

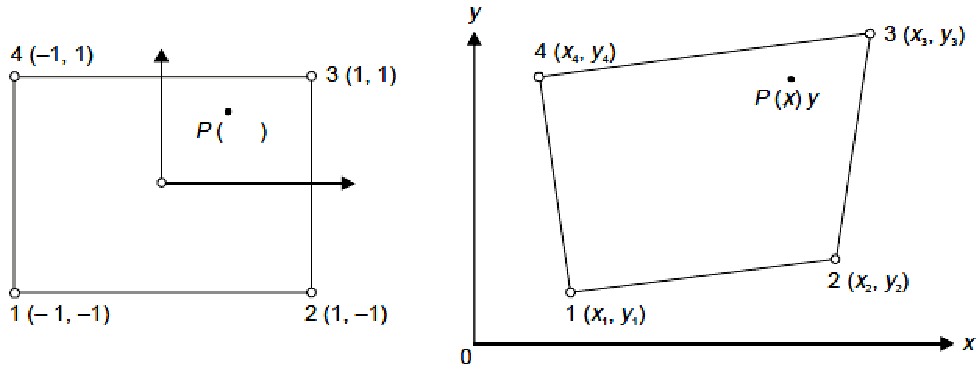
8. State the properties of a stiffness matrix.*(April/May 2015), (Nov/Dec 2012)*

The properties of the stiffness matrix [K] are,

1. It is a symmetric matrix
2. The sum of the elements in any column must be equal to zero.
3. It is an unstable element, so the determinant is equal to zero.

9. Show the transformation for mapping x-coordinate system into a natural coordinate system for a linear bar element and a quadratic bar element.*(Nov/Dec 2012)*

For example consider mapping of a rectangular parent element into a quadrilateral element



(a) Parent element (b) Mapped element

The shape functions of this element are

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

P is a point with coordinate (ξ, η). In global system the coordinates of the nodal points are

To get this mapping we define the coordinate of point P as,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

10. Define dynamic analysis.*(May/June 2014)*

When the inertia effect due to the mass of the components is also considered in addition to the externally applied load, then the analysis is called dynamic analysis.

11. What are the types of boundary conditions used in one dimensional heat transfer problems?

(i) Imposed temperature

(ii) Imposed heat flux

(iii) Convection through an end node.

12. What are the difference between boundary value problem and initial value problem?

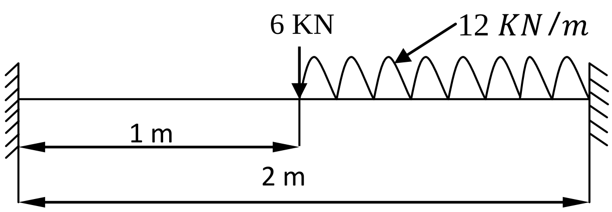
(i) The solution of differential equation obtained for physical problems which satisfies some specified conditions known as boundary conditions.

(ii) If the solution of differential equation is obtained together with initial conditions then it is known as initial value problem.

(iii) If the solution of differential equation is obtained together with boundary conditions then it is known as boundary value problem.

PART -B

1. For the beam and loading shown in fig. calculate the nodal displacements. Take [E] =210 GPa =210×10^9 N/m^2 , [I] = 6×10^{-6} m^4 NOV / DEC 2013



Given data

Young's modulus [E] =210 GPa =210×10^9 N/m^2

Moment of inertia [I] = 6×10^{-6} m^4

Length $[L]_1$ = 1m

Length $[L]_2$ = 1m

W=12 kN/m =12×10^3 N/m

F = 6KN

To find

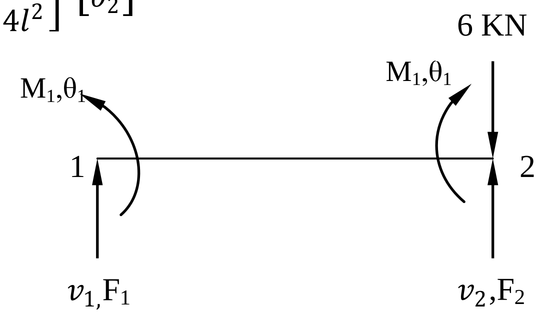
- Deflection

Formula used

$$f(x)\begin{bmatrix} \frac{-l}{2} \\ \frac{-l^2}{12} \\ \frac{-l}{2} \\ \frac{l^2}{12} \end{bmatrix} + \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix}$$

Solution

For element 1



$$f(x)\begin{bmatrix}\frac{-l}{2}\\ \frac{-l^2}{12}\\ \frac{-l}{2}\\ \frac{l^2}{12}\end{bmatrix}+\begin{bmatrix}F_1\\ M_1\\ F_2\\ M_2\end{bmatrix}=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix}$$

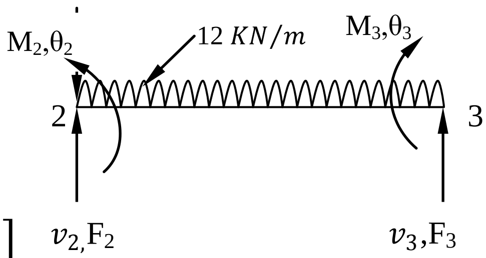
Applying boundary conditions

F_1=0N ; F_2=-6KN=-6×10^3 N; f(x)=0

M_1=M_2=0; u_1=0; θ_1=0; $u_2\neq0$; $\theta_2\neq0$

$$10^3\times\begin{bmatrix}0\\ 0\\ -6\\ 0\end{bmatrix}=\frac{210\times10^9\times6\times10^{-6}}{1^3}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{bmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix}$$

$$=1.26\times10^6\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_2\\ 0\end{Bmatrix}$$



For element 2

$$f(x)\begin{bmatrix}\frac{-l}{2}\\ \frac{-l^2}{12}\\ \frac{-l}{2}\\ \frac{l^2}{12}\end{bmatrix}+\begin{bmatrix}F_2\\ M_2\\ F_3\\ M_3\end{bmatrix}=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}u_2\\ \theta_2\\ u_3\\ \theta_3\end{bmatrix}$$

Applying boundary conditions

f(x) = -12 kN/m =12×10^3 N/m; $F_2=F_3=0=M_2=M$;

$u_2\neq0$; $\theta_2\neq0$; $u_3=\theta_3=0$

$$10^3\times\begin{Bmatrix}-6\\ -1\\ -6\\ 1\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}=1.26\times10^6\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

$$10^3\times\begin{Bmatrix}-6\\ -1\\ -6\\ 1\end{Bmatrix}=1.26\times10^6\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

Assembling global matrix

$$10^3\times\begin{Bmatrix}0\\0\\-12\\-1\\-6\\1\end{Bmatrix} = 1.26\times10^6\times\begin{bmatrix}12 & 6 & -12 & -6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\0\\u_2\\\theta_2\\0\\0\end{Bmatrix}$$

Solving matrix

$-12\times10^3=1.26\times10^6\times24u_2=0;$ $u_2=-3.96\times10^{-4}$m

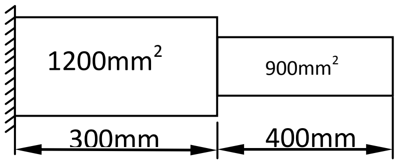
$-1\times10^3=1.26\times10^6\times8\theta_2=0;$ $\theta_2=-9.92$rad

Result

$\theta_2=-9.92$rad

$u_2=-3.96\times10^{-4}$m

2. Determine the axial vibration of a steel bar shown in fig. Take [E] =2.1×10^5 N/mm^2, [ρ] = 7800 kg/m^3 NOV/DEC 2014



Given data

$A_1=1200mm^2$; $A_2=900mm^2$

$l_1=300mm$; $l_2=400mm$

Young's modulus [E] $=2.1\times10^5\ N/mm^2$

Density [ρ] $= 7800\ Kg/m^3$

$=7.8\times10^{-6}\ Kg/mm^3$

To find

- Stiffness matrix
- Mass matrix
- Natural frequency
- Mode shape

Formula used

General equation for free vibration of bar $|k-m\lambda|$ {u}= 0

Stiffness matrix $[k] = \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

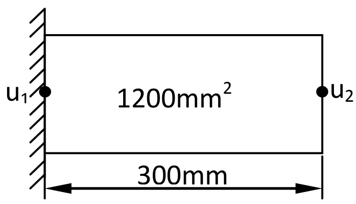
Consistent mass matrix $[m] = \frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

Lumped mass matrix $[m] = \frac{\rho AL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Mode shape $|k-m\lambda|U_1 = 0$; Normalization $U_1^T M\ U_1 = 1$

Solution

For element 1



Stiffness matrix [k] = $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$[k_1] = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $\quad = \frac{1200\times2.1\times10^5}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= 8.4\times10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $\quad = 10^5\begin{bmatrix} 8.4 & -8.4 \\ -8.4 & 8.4 \end{bmatrix}$

Consistent mass matrix [m] = $\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$;

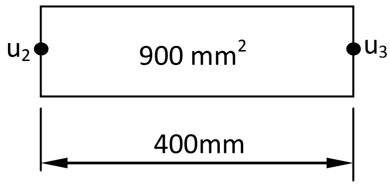
$$[m_1] = \frac{\rho A_1L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1200\times300\times7.8\times10^{-6}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.468\times\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_1] = \begin{bmatrix} 0.936 & 0.468 \\ 0.468 & 0.936 \end{bmatrix}$$

For element 2



Stiffness matrix [k] = $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[k_2] = \frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

$$= \frac{900\times2.1\times10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 4.73\times10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_2] = 10^5\begin{bmatrix} 4.73 & -4.73 \\ -4.73 & 4.73 \end{bmatrix};$$

Consistent mass matrix [m] = $\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$;

$$[m_2] = \frac{\rho A_2L_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{900\times400\times7.8\times10^{-6}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.468\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_2] = \begin{bmatrix} 0.936 & 0.468 \\ 0.468 & 0.936 \end{bmatrix}$$

Assembling global matrix

Stiffness matrix [k] = $10^5\begin{bmatrix} 8.4 & -8.4 & 0 \\ -8.4 & 13.13 & -4.73 \\ 0 & -4.73 & 4.73 \end{bmatrix}$

Consistent mass matrix [m] = $\begin{bmatrix} 0.936 & 0.468 & 0 \\ 0.468 & 1.87 & 0.468 \\ 0 & 0.468 & 0.936 \end{bmatrix}$

General equation for free vibration of bar $|k - m\lambda|$ {u} = 0

$$10^5 \begin{bmatrix} 8.4 & -8.4 & 0 \\ -8.4 & 13.13 & -4.73 \\ 0 & -4.73 & 4.73 \end{bmatrix} - \lambda \begin{bmatrix} 0.936 & 0.468 & 0 \\ 0.468 & 1.87 & 0.468 \\ 0 & 0.468 & 0.936 \end{bmatrix} = 0$$

$$10^5 \begin{bmatrix} 13.13 & -4.73 \\ -4.73 & 4.73 \end{bmatrix} - \lambda \begin{bmatrix} 1.87 & 0.468 \\ 0.468 & 0.936 \end{bmatrix} = 0$$

$$\begin{bmatrix} 13.13 \times 10^5 - 1.87\lambda & -4.73 \times 10^5 - 0.468\lambda \\ -4.73 \times 10^5 - 0.468\lambda & 4.73 \times 10^5 - 0.936\lambda \end{bmatrix} = 0$$

$$[(13.13\times10^5 - 1.87\lambda)(4.73 \times 10^5 - 0.936\lambda) - (-4.73 \times 10^5 - 0.468\lambda)(-4.73 \times 10^5 - 0.468\lambda)] = 0$$

$$6.2\times10^{11} - 1.23\times 10^6 \lambda - 8.84\times10^5 \lambda + 1.75\times\lambda^2 - 2.24\times10^{11} - 2.21\times10^5 \lambda - 2.21\times10^5 \lambda - 0.22 \lambda^2 = 0$$

$$1.53\lambda^2 - 2.55\times10^5 \lambda + 3.96\times10^{11} = 0$$

Solving above equation

$\lambda_1 = 1.49\times10^6$

$\lambda_2 = 1.73\times10^5 = 0.173\times10^6$

To find mode shape

$|k - m\lambda|\{u\} = 0$;

$\lambda_1 = 0.173\times10^6$

$$10^5 \begin{bmatrix} 13.13 & -4.73 \\ -4.73 & 4.73 \end{bmatrix} - 0.173\times10^6 \begin{bmatrix} 1.87 & 0.468 \\ 0.468 & 0.936 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 0.99 \times 10^6 & -0.55 \times 10^6 \\ -0.55 \times 10^6 & 0.31 \times 10^6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0$$

$0.99\times10^6 u_2 - 0.55\times 10^6 u_3 = 0$

$-0.55\times10^6 u_2 + 0.31\times10^6 u_3 = 0$

$u_3 = 1.77u_2$

$|k - m\lambda|\{u\} = 0$

$\lambda_2 = 1.49\times10^6$

$$10^5 \begin{bmatrix} 13.13 & -4.73 \\ -4.73 & 4.73 \end{bmatrix} - 1.49\times10^6 \begin{bmatrix} 1.87 & 0.468 \\ 0.468 & 0.936 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} -1.48 \times 10^6 & -1.17 \times 10^6 \\ -1.17 \times 10^6 & -0.924 \times 10^6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0$$

$-1.482\times10^6 u_2 - 1.17\times 10^6 u_3 = 0$

$-1.17\times10^6 u_2 - 0.924\times10^6 u_3 = 0$

$u_3 = -1.26u_2$

Normalization $U_1^T M U_1 = 1$

Normalization of λ_1

$$[u_2 \quad 1.77u_2]\begin{bmatrix}1.87 & 0.468\\0.46 & 0.936\end{bmatrix}\begin{Bmatrix}u_2\\1.77u_2\end{Bmatrix}=1$$

$$[u_2 \quad 1.77u_2]\begin{Bmatrix}u_2\\1.77u_2\end{Bmatrix}=1$$

$$2.7u_2^2 + 3.79u_2^2 = 1$$

$$u_2^2 = \frac{1}{6.4}; \qquad u_2 = 0.392$$

$$u_3 = 1.78u_2; \qquad u_3 = 0.698$$

Normalization of λ_2

$$U_2^T \text{M } \text{U}_2 = 1$$

$$[u_2 \quad -1.26u_2]\begin{bmatrix}1.87 & 0.468\\0.46 & 0.936\end{bmatrix}\begin{Bmatrix}u_2\\-1.26u_2\end{Bmatrix}=1$$

$$[1.28u_2 \quad -0.707u_2]\begin{Bmatrix}u_2\\-1.256u_2\end{Bmatrix}=1$$

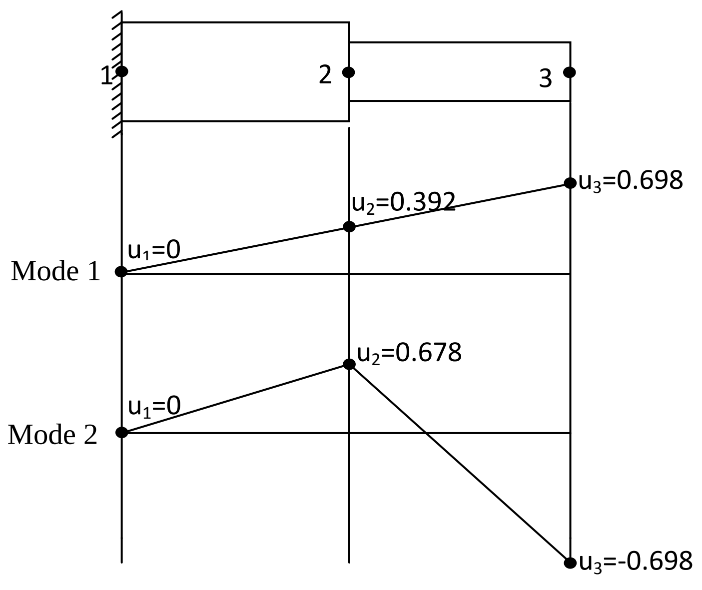
$$1.28u_2^2 + 0.88u_2^2 = 1$$

$$u_2^2 = 0.46; \qquad u_3 = -1.268u_2$$

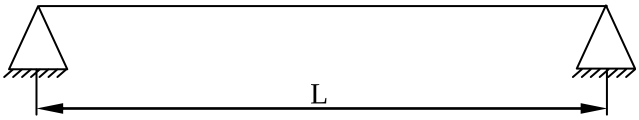
$$u_3 = -0.84$$

Result

Mode shape



3. **Consider the simply supported beam shown in fig. let the length L=1m, E=2×10^11 N/m^2, area of cross section A=30cm², moment of inertia I=100mm⁴, density[ρ] = 7800kg/m^3. Determine the natural frequency using two types of mass matrix. Lumped mass matrix and consistent mass matrix.** **APRIL / MAY 2011**



Given data

Length = 1m

Young's modulus E=2×10^{11} N/m^2

Area A=30cm^2 = 3×10^{-3} m^2

Moment of inertia I=100mm^4 = 100×10^{-12} m^4

Density[ρ] = 7800 kg/m^3=76518 N/m^3

To find

- Lumped mass matrix
- Consistent mass matrix
- Natural frequency

Formula used

General equation for free vibration of beam $|k - \omega^2 m|$ {u} = 0

Stiffness matrix[k] = $\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

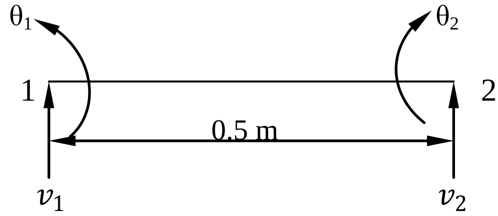
Consistent mass matrix [m] = $\frac{\rho AL}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$

Lumped mass matrix [m] = $\frac{\rho Al}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

For element 1

Stiffness matrix$[k]_1 = \frac{E_1 I}{l_1^3}\begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 \\ 6l_1 & 4l_1^2 & -6l_1 & 2l_1^2 \\ -12 & -6l_1 & 12 & -6l_1 \\ 6l_1 & 2l_1^2 & -6l_1 & 4l_1^2 \end{bmatrix}$



$$=\frac{2\times10^{11}\times100\times^{-12}}{0.5^3}\begin{bmatrix} 12 & 6\times0.5 & -12 & 6\times0.5 \\ 6\times0.5 & 4\times0.5^2 & -6\times0.5 & 2\times0.5^2 \\ -12 & -6\times0.5 & 12 & -6\times0.5 \\ 6\times0.5 & 2\times0.5^2 & -6\times0.5 & 4\times0.5^2 \end{bmatrix}$$

$$[k]_1 = 160\times\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$$

Lumped mass matrix $[m]_1 = \frac{\rho A l_1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$=\frac{76518\times3\times10^{-3}\times0.5}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[m]_1 = \begin{bmatrix} 57.38 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 57.38 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

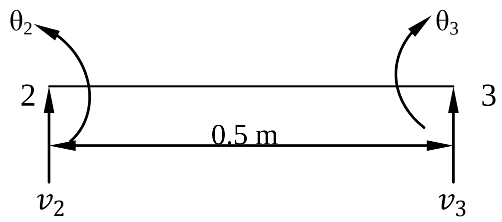
Consistent mass matrix $[m]_1 = \frac{\rho A l_1}{420}\begin{bmatrix} 156 & 22l_1 & 54 & -13l_1 \\ 22l_1 & 4l_1^2 & 13l_1 & -3l_1^2 \\ 54 & 13l_1 & 156 & -22l_1 \\ -13l_1 & -3l_1^2 & -22l_1 & 4l_1^2 \end{bmatrix}$

$$=\frac{76518\times3\times10^{-3}\times0.5}{420}\begin{bmatrix} 156 & 22\times0.5 & 54 & -13\times0.5 \\ 22\times0.5 & 4\times0.5^2 & 13\times0.5 & -3\times0.5^2 \\ 54 & 13\times0.5 & 156 & -22\times0.5 \\ -13\times0.5 & -3\times0.5^2 & -22\times0.5 & 4\times0.5^2 \end{bmatrix}$$

$$[m]_1 = \begin{bmatrix} 42.63 & 3 & 14.74 & -1.77 \\ 3 & 0.27 & 1.77 & -0.20 \\ 14.74 & 1.77 & 42.63 & -3 \\ -1.77 & -0.20 & -3 & 0.27 \end{bmatrix}$$

For element 2

Stiffness matrix $[k]_2 = \frac{EI}{l_2^3}\begin{bmatrix} 12 & 6l_2 & -12 & 6l_2 \\ 6l_2 & 4l_2^2 & -6l_2 & 2l_2^2 \\ -12 & -6l_2 & 12 & -6l_2 \\ 6l_2 & 2l_2^2 & -6l_2 & 4l_2^2 \end{bmatrix}$



$$=\frac{2\times10^{11}\times100\times^{-12}}{0.5^3}\begin{bmatrix} 12 & 6\times0.5 & -12 & 6\times0.5 \\ 6\times0.5 & 4\times0.5^2 & -6\times0.5 & 2\times0.5^2 \\ -12 & -6\times0.5 & 12 & -6\times0.5 \\ 6\times0.5 & 2\times0.5^2 & -6\times0.5 & 4\times0.5^2 \end{bmatrix}$$

$$[k]_2 = 160\times\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$$

Lumped mass matrix $[m]_2 = \frac{\rho A l_2}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$=\frac{76518\times3\times10^{-3}\times0.5}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[m]_2 = \begin{bmatrix} 57.38 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 57.38 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent mass matrix $[m]_2 = \frac{\rho A l_2}{420}\begin{bmatrix} 156 & 22l_2 & 54 & -13l_2 \\ 22l_2 & 4l_2^2 & 13l_2 & -3l_2^2 \\ 54 & 13l_2 & 156 & -22l_2 \\ -13l_2 & -3l_2^2 & -22l_2 & 4l_2^2 \end{bmatrix}$

$$=\frac{76518\times3\times10^{-3}\times0.5}{420}\begin{bmatrix} 156 & 22\times0.5 & 54 & -13\times0.5 \\ 22\times0.5 & 4\times0.5^2 & 13\times0.5 & -3\times0.5^2 \\ 54 & 13\times0.5 & 156 & -22\times0.5 \\ -13\times0.5 & -3\times0.5^2 & -22\times0.5 & 4\times0.5^2 \end{bmatrix}$$

$$[m]_2 = \begin{bmatrix} 42.63 & 3 & 14.74 & -1.77 \\ 3 & 0.27 & 1.77 & -0.20 \\ 14.74 & 1.77 & 42.63 & -3 \\ -1.77 & -0.20 & -3 & 0.27 \end{bmatrix}$$

Global matrix

Stiffness matrix [k] =160× $\begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix}$

Lumped mass matrix [m]= $\begin{bmatrix} 57.38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 114.77 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.38 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Consistent mass matrix[m]= $\begin{bmatrix} 42.63 & 3 & 14.74 & -1.77 & 0 & 0 \\ 3 & 0.27 & 1.77 & -0.2 & 0 & 0 \\ 14.74 & 1.77 & 85.26 & 0 & 14.74 & -1.77 \\ -1.77 & -0.2 & 0 & 0.5 & 1.77 & -0.2 \\ 0 & 0 & 14.74 & 1.77 & 42.63 & -3 \\ 0 & 0 & -1.77 & -0.2 & -3 & 0.27 \end{bmatrix}$

Frequency for lumped mass matrix

$$|k - \omega^2 m|\{u\} = 0$$

$$\left| 160 \times \begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 57.38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 114.77 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.38 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right| \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = 0$$

Applying boundary conditions

$v_1=0=\theta_1$; $v_2\neq0$; $\theta_2\neq0$ $v_3=0=\theta_3$;

$$\left| 160 \times \begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 57.38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 114.77 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.38 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right| \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix} = 0$$

$$\left| 160 \times \begin{bmatrix} 24 & 0 \\ 0 & 2 \end{bmatrix} - \omega^2 \begin{bmatrix} 114.7 & 0 \\ 0 & 0 \end{bmatrix} \right| \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 3840 - \omega^2 \times 114.7 & 0 - 0 \\ 0 - 0 & 320 - 0 \end{bmatrix} = 0$$

$\{(3840 - \omega^2 \times 114.7) \times (320 - 0)\text{-}0\text{-}0\} = 0$

$1228800\text{-}36704\omega^2 = 0$

$\omega^2 = 33.47$

$\omega = 5.78\ {}^{rad}/_{s}$

Frequency for consistent mass matrix

$$|k - \omega^2 m|\{u\} = 0$$

$$\left| 160 \times \begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 42.63 & 3 & 14.74 & -1.77 & 0 & 0 \\ 3 & 0.27 & 1.77 & -0.2 & 0 & 0 \\ 14.74 & 1.77 & 85.26 & 0 & 14.74 & -1.77 \\ -1.77 & -0.2 & 0 & 0.5 & 1.77 & -0.2 \\ 0 & 0 & 14.74 & 1.77 & 42.63 & -3 \\ 0 & 0 & -1.77 & -0.2 & -3 & 0.27 \end{bmatrix} \right| \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = 0$$

Applying boundary conditions

v_1=0=θ_1; v_2≠0; θ_2≠0 v_3=0=θ_3;

$$\left| 160 \times \begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 42.63 & 3 & 14.74 & -1.77 & 0 & 0 \\ 3 & 0.27 & 1.77 & -0.2 & 0 & 0 \\ 14.74 & 1.77 & 85.26 & 0 & 14.74 & -1.77 \\ -1.77 & -0.2 & 0 & 0.5 & 1.77 & -0.2 \\ 0 & 0 & 14.74 & 1.77 & 42.63 & -3 \\ 0 & 0 & -1.77 & -0.2 & -3 & 0.27 \end{bmatrix} \right| \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix} = 0$$

$$160 \times \left| \begin{bmatrix} 24 & 0 \\ 0 & 2 \end{bmatrix} - \omega^2 \begin{bmatrix} 85.26 & 0 \\ 0 & 0.5 \end{bmatrix} \right| \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 3840 - 85.26\omega^2 & 0 - 0 \\ 0 - 0 & 320 - 0.5\omega^2 \end{bmatrix} = 0$$

$$(3840 - 85.26\omega^2)(320 - 0.5\omega^2) = 0$$

$$1.23\times10^6 - 1920\omega^2 - 27283.2\omega^2 + 42.63\omega^4 = 0$$

Take $\lambda = \omega^2$

$42.63\,\lambda^2 - 29203.3\,\lambda + 1.23\times10^6 = 0$ $ax^2 + bx + c = 0$; $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\lambda = \frac{29203.3 \pm \sqrt{29203.3^2 - 4\times42.63\times1.23\times10^6}}{2\times42.63}$$

$$= \frac{29203.3 \pm 25359.28}{85.26}$$

$\lambda_1 = \frac{29203.3 + 25359.28}{85.26}$; $\lambda_2 = \frac{29203.3 - 25359.28}{85.26}$

λ_1 =639.95; λ_2=45.08

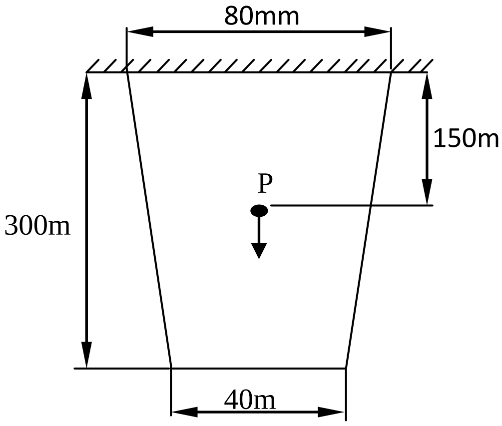
$\lambda = \omega^2$

$\omega_1 = \sqrt{\lambda_1}$; $\omega_2 = \sqrt{\lambda_2}$

$\omega_1 = \sqrt{639.95}$ $\omega_2 = \sqrt{45.08}$

ω_1= 25.3 rad/s ω_2= 6.7 rad/s

4. **For a tapered plate of uniform thickness t = 10mm as shown in fig. find the displacements at the nodes by forming in to two element model. The bar has mass density ρ = 7800Kg/m^3 Young's modulus E = 2×10^5MN/m^2. In addition to self weight the plate is subjected to a point load p = 10KN at its centre. Also determine the reaction force at the support.** **Nov/Dec 2006**



Given data

Mass density ρ = 7800kg/m^3

= 7800 × 9.81=76518 N/m^3

= 7.65 × 10^{-5} N/mm^3

Young's modulus E = 2×10^5MN/m^2;

= 2×10^5 × 10^6 N/m^2

= 2×10^5 N/mm^2

Point load P = 10 KN

To find

- Displacement at each node
- Reaction force at the support

Formula used

{F} =[K] {u}

Stiffness matrix [k] = $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

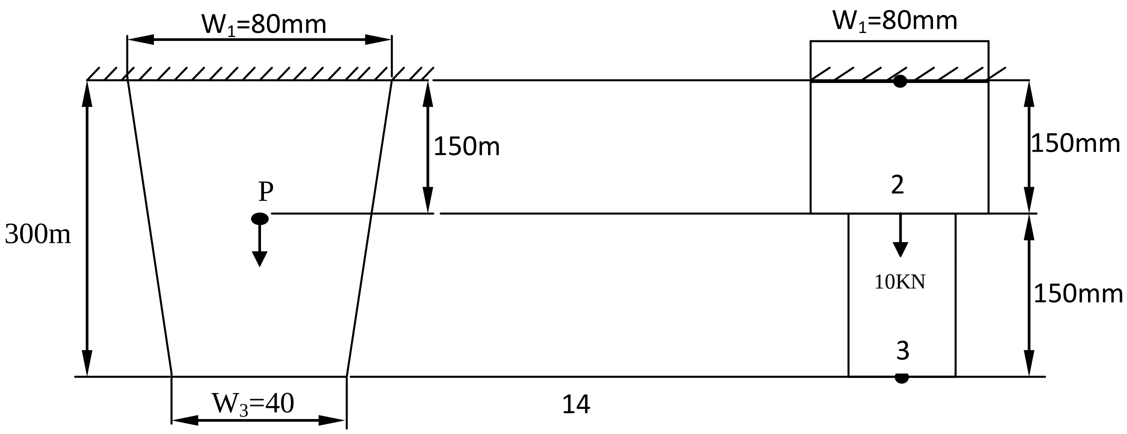
Force vector $\{F\}$ = $\frac{\rho Al}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

{R} =[K] {u} -{F}

Solution

The given taper bar is considered as stepped bar as shown in fig.



W_1 = 80mm

$W_2 = \frac{W_1+W_3}{2} = \frac{80+40}{2}$ = 60 mm

W_3 = 40mm

Area at node 1 A_1 = Width × thickness

$= W_1 \times t_1$

= 80 × 10 = 800mm^2

Area at node 2; A_2 = Width × thickness

$= W_2 \times t_2$ = 60 × 10 =600mm^2

Area at node 1 A_1 = Width × thickness

= $W_3 \times t_3$ = 40 × 10 =400mm^2

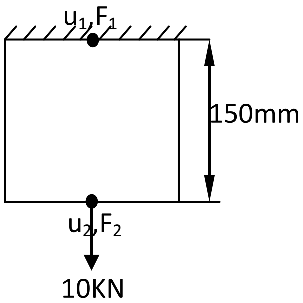
Average area of element 1

$\bar{A}_1 = \frac{Area\ of\ node\ 1 + Area\ of\ node\ 2}{2} = \frac{A_1 + A_2}{2} = \frac{800+600}{2}$ = 700mm^2

Average area of element 2

$\bar{A}_2 = \frac{Area\ of\ node\ 2 + Area\ of\ node\ 3}{2} = \frac{A_2 + A_3}{2} = \frac{600+400}{2}$ = 500mm^2

For element 1



Stiffness matrix $[k]_1 = \frac{\bar{A}_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

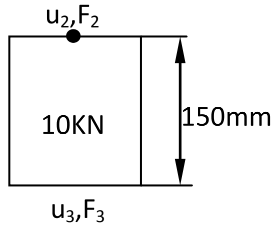
$= \frac{700 \times 2\times10^5}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$= 2\times 10^5 \begin{bmatrix} 4.67 & -4.67 \\ -4.67 & 4.67 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Force vector $\{F\}_1 = \frac{\rho \bar{A}_1 l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$= \frac{7.65\times10^{-5}\times700\times150}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 4.017 \end{Bmatrix}$

For element 2



Stiffness matrix $[k]_2 = \frac{\bar{A}_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

$= \frac{500 \times 2\times10^5}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

$= 2\times 10^5 \begin{bmatrix} 3.33 & -3.33 \\ -3.33 & 3.33 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

Force vector $\{F\}_2 = \frac{\rho \bar{A}_2 l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{7.65 \times 10^{-5} \times 500 \times 150}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad = \begin{Bmatrix} 2.869 \\ 2.869 \end{Bmatrix}$$

Global matrix

Stiffness matrix [k] = $2 \times 10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix}$

Force vector $\{F\} = \begin{Bmatrix} 4.017 \\ 6.88 \\ 2.87 \end{Bmatrix}$

Finite element equation

{F} =[K] {u}

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 2 \times 10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Applying boundary conditions

$u_1 = 0$; $u_2 \neq 0$; $u_3 \neq 0$; $F_2 = 10 \times 10^3$ N

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 2 \times 10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 4.017 \\ 6.88 + 10 \times 10^3 \\ 2.87 \end{Bmatrix} = 2 \times 10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 10006.88 \\ 2.86 \end{Bmatrix} = 2 \times 10^5 \begin{bmatrix} 7.99 & -3.33 \\ -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$2 \times 10^5 (7.99 u_2 - 3.33 u_3) = 10006.88$

$2 \times 10^5 (-3.33 u_2 + 3.33 u_3) = 2.86$

Solving above equation

$2 \times 10^5 (4.66\, u_2) = 10009.74$

$u_2 = 0.01074$ mm

$2 \times 10^5 (-3.33 \times 0.01074 + 3.33 u_3) = 2.86$

$666000 u_3 = 2.86 + 7152.88$

$u_3 = 0.01074$

Reaction force

$$\{R\} = [K]\{u\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 2\times10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 2\times10^5 \times \begin{bmatrix} 4.66 & -4.66 & 0 \\ -4.66 & 7.99 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0.01074 \\ 0.01074 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10006.88 \\ 2.87 \end{Bmatrix}$$

$$= 2\times10^5 \begin{Bmatrix} 0 - 0.05 + 0 \\ 0 + 0.086 - 0.036 \\ 0 - 0.036 + 0.036 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10006.88 \\ 2.87 \end{Bmatrix}$$

$$= 2\times10^5 \begin{Bmatrix} -0.05 \\ 0.05 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10006.88 \\ 2.87 \end{Bmatrix}$$

$$= \begin{Bmatrix} -10000 \\ 10000 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10006.88 \\ 2.87 \end{Bmatrix}$$

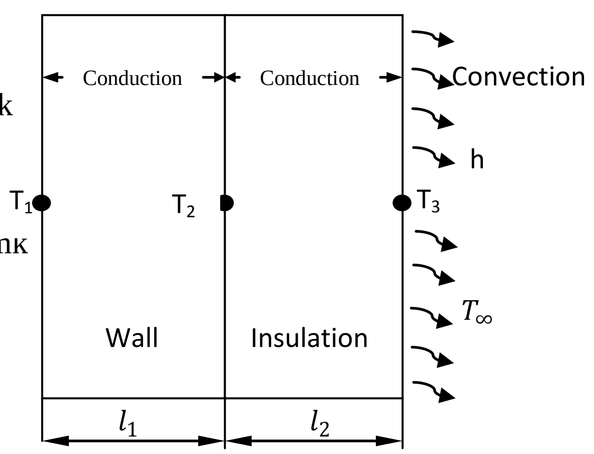
$$= \begin{Bmatrix} -10004.017 \\ -6.88 \\ -2.86 \end{Bmatrix}$$

Result

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -10004.017 \\ -6.88 \\ -2.86 \end{Bmatrix}$$

5. A wall of 0.6m thickness having thermal conductivity of 1.2 W/mk. The wall is to be insulated with a material of thickness 0.06m having an average thermal conductivity of 0.3 W/mk. The inner surface temperature in 1000^{O}C and outside of the insulation is exposed to atmospheric air at 30^{o}c with heat transfer co-efficient of 35 W/m^{2}k. Calculate the nodal temperature. NOV/DEC 2014

Given Data:-



Thickness of the wall, $l_1 = 0.6$m

Thermal conductivity of the wall $K_1 = 1.2$W/mk

Thickness of the insulation $l_2 = 0.06$m

Thermal Conductivity of the wall $K_2 = 0.3$W/mκ

Inner surface temperature $T_1 = 1000^{o}C + 273$

$= 1273$ K

Atmospheric air temperature $T_2 = 30 + 273$

$= 303$ K

Heat transfer co-efficient at outer side $h = 35 W/m^2k$

To find

Nodal temperature T_2 and T_3

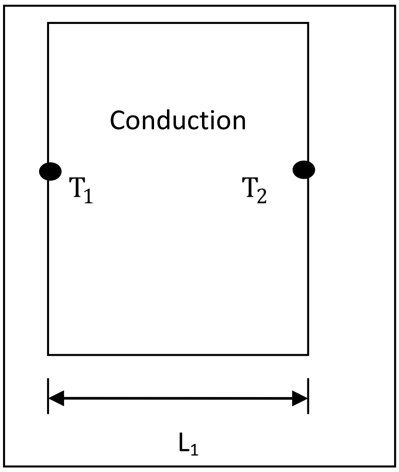
Formula used

1D Heat conduction

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

1D Heat conduction with free end convection

$$[K] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Solution

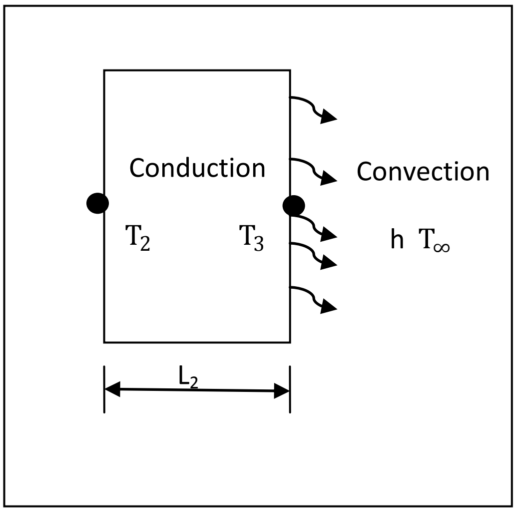
For element 1

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{k_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

For unit area: $A_1 = 1m^2$

$$= \frac{1.2}{0.6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$



For element (2)

$$\left\{\frac{A_2 K_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = h\, T_2 A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left\{\frac{1 \times 0.3}{0.06}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 35 \times 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = 35 \times 303 \times 1 \times \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left\{\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}\right\}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

$$\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Assembling finite element equation

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

Applying boundary conditions

$\{f_1\} = 0$

$f_2 = 0$
$f_3 = 10.605 \times 10^3$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step (1)

The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step – II

The first row of the force matrix is replaced by the known temperature at node 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step – III

The second row first column of stiffness [K] value is multiplied by known temperature at node 1 -2 × 1273 = -2546. This value positive digit 2546 has been added to the second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 2546 \\ 10.605 \times 10^3 \end{Bmatrix}$$

$\Longrightarrow \; 7\,T_2 - 5\,T_3 = 2546$
$-5\,T_2 + 40\,T_3 = 10.605 \times 10^3$

Solving above Eqn ×8 $\quad 56\,T_2 - 40T_3 = 20.368 \times 10^3$

$\quad 5\,T_2 - 40T_3 = 10.605 \times 10^3$

$$51\,T_2 = 30973$$

$$T_2 = 607.31 \text{ K}$$

$$7 \times 607.31 - 5\,T_3 = 2546$$

$$4251.19 - 5\,T_3 = 2546$$

$$-{-5}\,T_3 = -1705$$

$$T_3 = 341.03 \text{ K}$$

Result

Nodal Temp $T_1 = 1273$ K

$T_2 = 607.31$K

$T_3 = 341.03$ K

7. Derivation of the displacement function u and shape function N for one dimensional linear bar element. OR

Derive the shape function, stiffness matrix and load vector for one dimensional bar element. May / June 2013

Consider a bar with element with nodes 1 and 2 as shown in Fig. v_1 and v_2 are the displacement at the respective nodes. v_1 And v_2 is degree of freedom of this bar element.

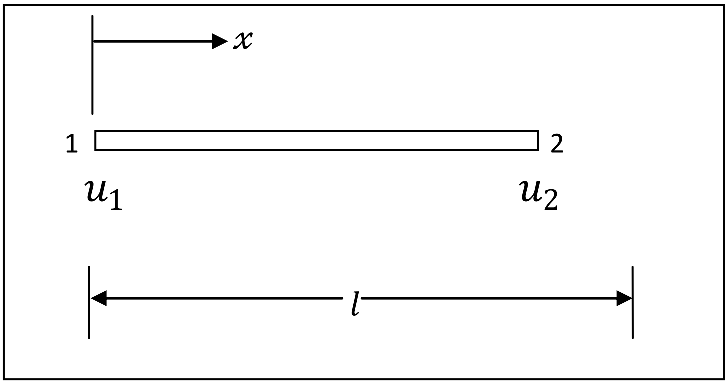


Fig Two node bar element

Since the element has got two degrees of freedom, it will have two generalized co-ordinates.

$$u = a_0 + a_1 x$$

Where, a_0 and a_1 are global or generalized co – ordinates.
Writing the equation in matrix form,

$$u = [1\ x] \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

At node 1, $u = u_1, x = 0$
At node 1, $u = u_2, x = 1$
Substitute the above values ion equation,

$$u_1 = a_0$$
$$u_2 = a_0 + a_1\, l$$

Arranging the equation in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$u^* \qquad C \qquad A$$

Where, $u^* \longrightarrow$ Degree of freedom.
$C \longrightarrow$ Connectivity matrix.
$A \longrightarrow$ Generalized or global co-ordinates matrix.

$$\Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l-0} \begin{bmatrix} 1 & -0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left[Note: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}\, a_{22} - a_{12} a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ $values\ in\ equation$

$$u = [1\ \ x]\frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l}\ [1\ \ x]\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l}\ [1 - x \qquad 0 + x]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$[\because Matrix\ Multiplication\ (1 \times 2)(2 \times 2) = (1 \times 2)]$

$$u = \begin{bmatrix} \frac{1-x}{l} & \frac{x}{l} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

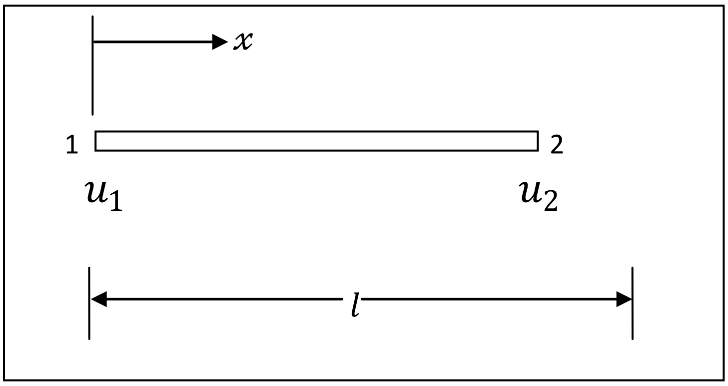
$$u = [N_1\ N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement function, $u = N_1\, u_1 + N_2\, u_2$

Where, Shape function, $N_1 = \frac{l-x}{l}$; $shape\ function, N_2 = \frac{x}{l}$

Stiffness matrix for one dimensional linear bar element

Consider a bar with element with nodes 1 and 2 as shown in Fig. v_1 and v_2 are the displacement at the respective nodes. v_1 And v_2 is degree of freedom of this bar element.



Stiffness matrix, $[K] = \int_v [\mathrm{B}]^{\mathrm{T}}[D][B]\, dv$

Displacement function, $u = N_1\, u_1 + N_2\, u_2$

Shape function, $N_1 = \frac{l-x}{l}$; $shape\ function, N_2 = \frac{x}{l}$

Strain displacement matrix,[B] = $\begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$

$$= \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$[\mathrm{B}]^{\mathrm{T}} = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

One dimensional problem [D] = [E] = young’s modulus

$$\mathbf{[K]} = \int_{\mathbf{0}}^{l} \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times E \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$=\int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \times E \times dv$$ [dv = A×dx

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \times E \times A \times dx$$

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \times \int_0^l dx = AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0)$$

$$= AE\ l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$=\frac{AEl}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Finite element equation for finite element analysis

$$\{F\} = [K]\ \{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Load vector [F]

Consider a vertically hanging bar of length$l$, uniform cross section A, density ρ and young's modulus E. this bar is subjected to self weight X_b

The element nodal force vector

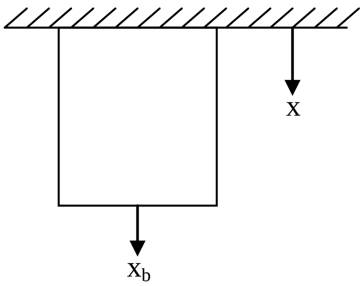
$$\{F\}_e = \int [N]^T\ X_b$$

Self weight due to loading force X_b = ρAdx

Displacement function, $u = N_1\, u_1 + N_2\, u_2$

Where; $N_1 = \frac{l - x}{l}$; $N_2 = \frac{x}{l}$;

$$[N] = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$



$$[N]^T = \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix}$$

Substitute X_b and $[N]^T$ values

$$\{F\}_e = \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \rho A \, dx = \rho A \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= \rho A \begin{Bmatrix} x - \frac{x^2}{2l} \\ \frac{x^2}{2l} \end{Bmatrix}_0^l = \rho A \begin{Bmatrix} l - \frac{l^2}{2l} \\ \frac{l^2}{2l} \end{Bmatrix} = \rho A \begin{Bmatrix} l - \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$= \rho A \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

Force vector $\{F\} = \frac{\rho A l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

7. DERIVATION OF SHAPE FUNCTION AN STIFFNESS MATRIX FOR ONE-DIMENSIONAL QUADRATIC BAR ELEMENT: May / June 2012

Consider a quadratic bar element with nodes 1,2 and 3 as shown in Fig.(i), u_1, u_2 and u_3 are the displacement at the respective nodes. So, u_1, u_2 and u_3 are considered as degree of freedom of this quadratic bar element.

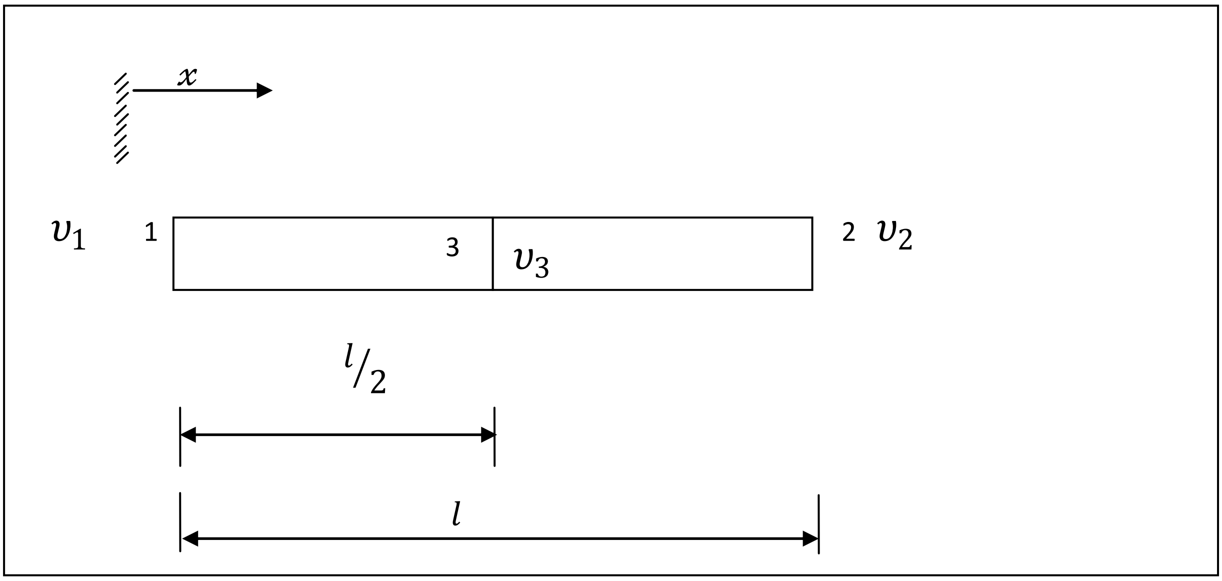


Fig. (i). Quadratic bar element

Since the element has got three nodal displacements, it will have three generalized coordinates.

$$u = a_0 + a_1 x + a_2 x^2$$

Where, a_0, a_1 and a_2 are global or generalized coordinates. Writing the equation is matrix form,

$$U = [1\ x\ \ x^2]\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}$$

$$\text{At node 1,} \quad \text{u} = \text{u}_1, \ \ x = 0$$

$$\text{At node 2,} \quad \text{u} = \text{u}_2, \ \ x = 1$$

$$\text{At node 3,} \quad \text{u} = \text{u}_3, \ \ x = \frac{1}{2}$$

Substitute the above values in equation.

$$\text{u}_1 = a_0$$

$$\text{u}_2 = a_0 + a_1\, l + a_2\, l^2$$

$$\text{u}_3 = a_0 + a_1\left(\frac{l}{2}\right) + a_2\left(\frac{l}{2}\right)^2$$

Substitute the equation we get

$$\Rightarrow \qquad \text{u}_2 = u_1 + a_1\, l + a_2\, l^2$$

$$\Rightarrow \qquad \text{u}_3 = u_1 + \frac{a_1\, l}{2} + \frac{a_2\, l^2}{4}$$

$$\Rightarrow \qquad \text{u}_2 - \text{u}_1 = a_1\, l + a_2\, l^2$$

$$\Rightarrow \qquad \text{u}_3 - u_1 = \frac{a_1\, l}{2} + \frac{a_2\, l^2}{4}$$

Arranging the equation in matrix form,

$$\begin{Bmatrix} \text{u}_2 - \text{u}_1 \\ \text{u}_3 - u_1 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{Bmatrix} \text{a}_1 \\ \text{a}_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \text{a}_1 \\ \text{a}_2 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix}^{-1} \begin{Bmatrix} \text{u}_2 - \text{u}_1 \\ \text{u}_3 - u_1 \end{Bmatrix}$$

$$= \frac{1}{\left(\frac{l^3}{4} - \frac{l^3}{2}\right)} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{2} & l \end{bmatrix} \begin{Bmatrix} \text{u}_2 - \text{u}_1 \\ \text{u}_3 - u_1 \end{Bmatrix}$$

$$\left[Note \ \because \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \ \text{X} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \begin{Bmatrix} \text{a}_1 \\ \text{a}_2 \end{Bmatrix} = \frac{1}{\left(\frac{-l^3}{4}\right)} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{2} & l \end{bmatrix} \begin{Bmatrix} \text{u}_2 - \text{u}_1 \\ \text{u}_3 - u_1 \end{Bmatrix}$$

$$\Rightarrow \ a_1 = \frac{-4}{l^3}\left[\frac{l^2}{4}\,(\text{u}_2 - \text{u}_1) - l^2(\text{u}_3 - u_1)\right]$$

$$\Rightarrow \quad a_2 = \frac{-4}{l^3}\left[\frac{-l}{2}(u_2 - u_1) + l(u_3 - u_1)\right]$$

Equation $\Rightarrow a_1 = \frac{-4}{l^3}\left[\frac{l^2 u_2}{4} - \frac{l^2 u_1}{4} - l^2 u_3 + l^2 u_1\right]$

$$= \frac{-4l^2 u_2}{4l^3} + \frac{4l^2 u_1}{4l^3} + \frac{4l^2 u_3}{l^3} - \frac{4l^2 u_1}{l^3}$$

$$= \frac{-u_2}{l} + \frac{u_1}{l} + \frac{4u_3}{l} - \frac{4u_1}{l}$$

$$a_1 = \frac{-3u_1}{l} - \frac{u_2}{l} + \frac{4u_3}{l}$$

Equation $\Rightarrow$

$$a_2 = \frac{-4}{l^3}\left[\frac{-lu_2}{2} - \frac{l}{2}u_1 + lu_3 - lu_1\right]$$

$$= \frac{4l\,u_2}{2\,l^3} + \frac{4l}{2\,l^3}u_1 - \frac{4l}{l^3}u_3 + \frac{4l}{l^3}u_1$$

$$= \frac{2u_2}{l^2} - \frac{2}{l^2}u_1 - \frac{4}{l^2}u_3 + \frac{4}{l^2}u_1$$

$$a_2 = \frac{2}{l^2}u_1 + \frac{2u_2}{l^2} - \frac{4}{l^2}u_3$$

Arranging the equation in matrix form,

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Substitution the equation

$$\{u\} = [1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = \left[\left(1 - \frac{3}{l}x + \frac{2x^2}{l^2}\right)\left(\frac{-x}{l} + \frac{2x^2}{l^2}\right)\left(\frac{4x}{l} - \frac{4x^2}{l^2}\right)\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = N_1 u_1 + N_2 u_2 + N_3 u_3$$

Where, shape function,

$$N_1 = 1 - \frac{3x}{l} + \frac{2x^2}{l^2}$$

$$N_2 = \frac{-x}{l} + \frac{2x^2}{l^2}$$

$$N_3 = \frac{4x}{l} - \frac{4x^2}{l^2}$$

STIFFNESS MATRIX FOR ONE-DIMENSIONAL QUADRATIC BAR ELEMENT:

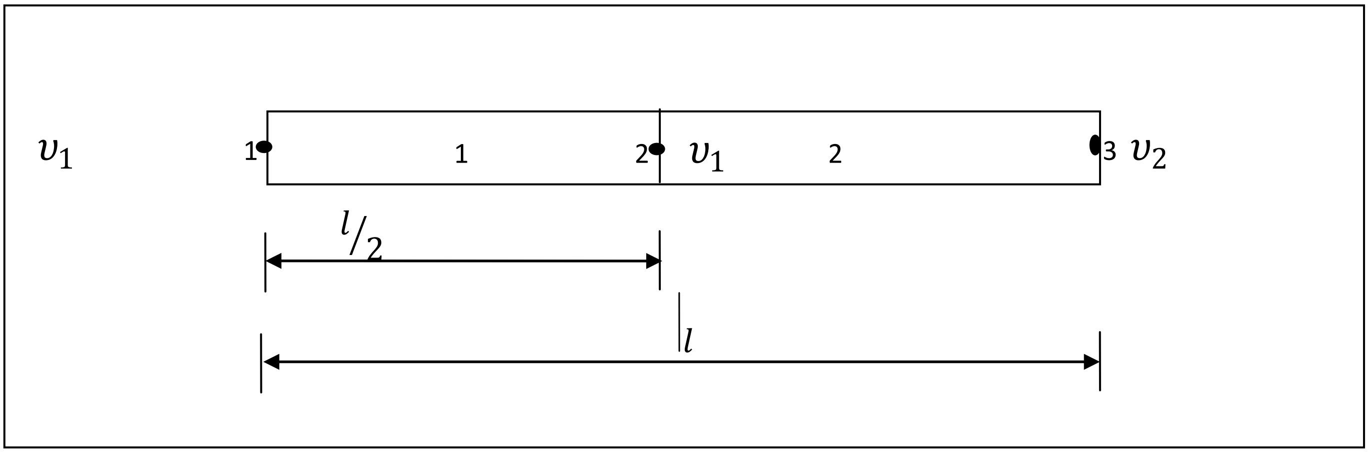


Fig. A bar element with three nodes

Consider a one dimensional quadratic bar element with nodes 1,2, and 3 as shown in Fig. 2. Let u_1, u_2 *and* u_3 be the nodal displacement parameters or otherwise known as degree of freedom.

We know that,

$$\text{Stiffness matrix, } [k] = \int_v [B]^T [D] [B] \, dv$$

In one dimensional quadratic bar element,

$$\text{Displacement function, } u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$\text{Where, } N_1 = 1 - \frac{3x}{l} + \frac{2x^2}{l^2}$$

$$N_2 = \frac{-x}{l} + \frac{2x^2}{l^2}$$

$$N_3 = \frac{4x}{l} - \frac{4x^2}{l^2}$$

We know that,

$$\text{Strain – Displacement matrix, } [B] = \left[\frac{d N_1}{dx} \; \frac{d N_2}{dx} \; \frac{d N_3}{dx} \right]$$

$$\Rightarrow \frac{d N_1}{dx} = \frac{-3}{l} + \frac{4x}{l^2}$$

$$\Rightarrow \frac{d N_2}{dx} = \frac{-1}{l} + \frac{4x}{l^2}$$

$$\Rightarrow \frac{d N_3}{dx} = \frac{4}{l} + \frac{8x}{l^2}$$

Substitute the equation

$$[B] = \left[\left(\frac{-3}{l} + \frac{4x}{l^2} \right) \left(\frac{-1}{l} + \frac{4x}{l^2} \right) \left(\frac{4}{l} - \frac{8x}{l^2} \right) \right]$$

$$[B]^T = \begin{Bmatrix} \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \\ \left(\frac{-1}{l} + \frac{4x}{l^2}\right) \\ \left(\frac{4}{l} + \frac{8x}{l^2}\right) \end{Bmatrix}$$

In one dimensional problems,

$$[D] = [E] = E = Young'sModulus$$

Substitute $[B]$ $[B]^T$ and $[D]$ values in stiffness matrix equation [$Limit\ is\ 0\ to\ l$].

$$\Rightarrow = \int_0^l \begin{Bmatrix} \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \\ \left(\frac{-1}{l} + \frac{4x}{l^2}\right) \\ \left(\frac{4}{l} - \frac{8x}{l^2}\right) \end{Bmatrix} \left[\left(\frac{-3}{l} + \frac{4x}{l^2}\right)\left(\frac{-1}{l} + \frac{4x}{l^2}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right)\right] \times \mathrm{E}\, dv$$

$$\Rightarrow [k] = EA \int_0^l \begin{bmatrix} \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \left(\frac{-3}{l} + \frac{4x}{l^2}\right) & \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \left(\frac{-1}{l} + \frac{4x}{l^2}\right) & \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \left(\frac{4}{l} - \frac{8x}{l^2}\right) \\ \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \left(\frac{-1}{l} + \frac{4x}{l^2}\right) & \left(\frac{-1}{l} + \frac{4x}{l^2}\right) \left(\frac{-1}{l} + \frac{4x}{l^2}\right) & \left(\frac{-1}{l} + \frac{4x}{l^2}\right) \left(\frac{4}{l} - \frac{8x}{l^2}\right) \\ \left(\frac{-3}{l} + \frac{4x}{l^2}\right) \left(\frac{4}{l} - \frac{8x}{l^2}\right) & \left(\frac{-1}{l} + \frac{4x}{l^2}\right) \left(\frac{4}{l} - \frac{8x}{l^2}\right) & \left(\frac{4}{l} - \frac{8x}{l^2}\right) \left(\frac{4}{l} - \frac{8x}{l^2}\right) \end{bmatrix} dx$$

$$\Rightarrow [k] = EA$$

$$\int_0^l \begin{bmatrix} \left(\frac{9}{l^2} - \frac{12x}{l^3} - \frac{12x}{l^3} + \frac{16x^2}{l^4}\right) & \left(\frac{3}{l^2} - \frac{12x}{l^3} - \frac{4x}{l^3} + \frac{16x^2}{l^4}\right) & \left(\frac{-12}{l^2} + \frac{24x}{l^3} + \frac{16x}{l^3} - \frac{32x^2}{l^4}\right) \\ \left(\frac{3}{l^2} - \frac{12x}{l^3} - \frac{4x}{l^3} + \frac{16x^2}{l^4}\right) & \left(\frac{1}{l^2} - \frac{4x}{l^3} - \frac{4x}{l^3} + \frac{16x^2}{l^4}\right) & \left(\frac{-4}{l^2} + \frac{8x}{l^3} + \frac{16x}{l^3} - \frac{32x^2}{l^4}\right) \\ \left(\frac{-12}{l^2} + \frac{24x}{l^3} + \frac{16x}{l^3} - \frac{32x^2}{l^4}\right) & \left(\frac{-4}{l^2} + \frac{8x}{l^3} + \frac{16x}{l^3} - \frac{32x^2}{l^4}\right) & \left(\frac{16}{l^2} - \frac{32x}{l^3} - \frac{32x}{l^3} + \frac{64x^2}{l^4}\right) \end{bmatrix} dx$$

$$= EA \begin{bmatrix} \left(\frac{9x}{l^2} - \frac{12x^2}{2\,l^3} - \frac{12x^2}{2\,l^3} + \frac{16x^3}{3\,l^4}\right) & \left(\frac{3x}{l^2} - \frac{12x^2}{2\,l^3} - \frac{4x^2}{2\,l^3} + \frac{16x^3}{3\,l^4}\right) & \left(\frac{-12}{l^2} + \frac{24x^2}{2\,l^3} + \frac{16x^2}{2\,l^3} - \frac{32x^3}{3\,l^4}\right) \\ \left(\frac{3x}{l^2} - \frac{12x^2}{2\,l^3} - \frac{4x^2}{2\,l^3} + \frac{16x^3}{3\,l^4}\right) & \left(\frac{x}{l^2} - \frac{4x^2}{2\,l^3} - \frac{4x^2}{2\,l^3} + \frac{16x^2}{3\,l^4}\right) & \left(\frac{-4}{l^2} + \frac{8x^2}{2\,l^3} + \frac{16x^2}{2\,l^3} - \frac{32x^2}{3\,l^4}\right) \\ \left(\frac{-12}{l^2} + \frac{24x^2}{2\,l^3} + \frac{16x^2}{2\,l^3} - \frac{32x^2}{3\,l^4}\right) & \left(\frac{-4}{l^2} + \frac{8x^2}{2\,l^3} + \frac{16x^2}{2\,l^3} - \frac{32x^2}{3\,l^4}\right) & \left(\frac{16}{l^2} - \frac{32x^2}{2\,l^3} - \frac{32x}{2\,l^3} + \frac{64x^2}{3\,l^4}\right) \end{bmatrix} dx$$

$$\Rightarrow [k] = EA \begin{bmatrix} \left(\frac{9}{l} - \frac{6}{l} - \frac{6}{l} + \frac{16}{3\,l}\right) & \left(\frac{3}{l} - \frac{6}{l} - \frac{2}{l} + \frac{16}{3\,l}\right) & \left(\frac{-12}{l} + \frac{12}{l} + \frac{8}{l} - \frac{32}{3\,l}\right) \\ \left(\frac{3}{l} - \frac{6}{l} - \frac{2}{l} + \frac{16}{3\,l}\right) & \left(\frac{1}{l} - \frac{2}{l} - \frac{4}{l} + \frac{16}{l}\right) & \left(\frac{-4}{l} + \frac{4}{l} + \frac{8}{l} - \frac{32}{3\,l}\right) \\ \left(\frac{-12}{l} + \frac{12}{l} + \frac{8}{l} - \frac{32}{3\,l}\right) & \left(\frac{-4}{l^2} + \frac{4}{l} + \frac{8}{l} - \frac{32}{3\,l}\right) & \left(\frac{16}{l} - \frac{16}{l} - \frac{16}{l} + \frac{64}{3\,l}\right) \end{bmatrix}$$

$$\Rightarrow [k] = EA \begin{bmatrix} \frac{7}{3\,l} & \frac{1}{3\,l} & \frac{-8}{3\,l} \\ \frac{1}{3\,l} & \frac{7}{3\,l} & \frac{-8}{3\,l} \\ \frac{-8}{3\,l} & \frac{-8}{3\,l} & \frac{16}{3\,l} \end{bmatrix}$$

$$\Rightarrow [k] = \frac{EA}{3\,l} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

LOAD VECTOR FOR ONE DIMENSIONAL QUADRATIC BAR ELEMENT:

`We know that, general force vector is,

$$\{F\} = \int_0^l [N]^T \;\; X_b$$

Where, $\{N\}^T = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1 - \frac{3x}{l} + \frac{2x^2}{l^2} \\ \frac{-x}{l} + \frac{2x^2}{l^2} \\ \frac{4x}{l} - \frac{4x^2}{l^2} \end{Bmatrix}$

Due to self weight, $X_{b} = \rho$ A dx

Substitute the equation,

$$\{F\} = \int_0^l \begin{Bmatrix} 1 - \frac{3x}{l} + \frac{2x^2}{l^2} \\ \frac{-x}{l} + \frac{2x^2}{l^2} \\ \frac{4x}{l} - \frac{4x^2}{l^2} \end{Bmatrix} \rho \text{ A } dx$$

$$\{F\} = \rho \text{ A} \begin{Bmatrix} x - \frac{3x^2}{2\,l} + \frac{2x^3}{3\,l^2} \\ \frac{-x^2}{2\,l} + \frac{2x^3}{3\,l^2} \\ \frac{4x^2}{2\,l} - \frac{4x^3}{3\,l^2} \end{Bmatrix}_0^1$$

$$= \rho \text{ A} \begin{Bmatrix} 1 - \frac{3l^2}{2\,l} + \frac{2l^3}{3\,l^2} \\ \frac{-l^2}{2\,l} + \frac{2\,l^3}{3\,l^2} \\ \frac{4\,l^2}{2\,l} - \frac{4l^3}{3\,l^2} \end{Bmatrix}$$

$$= \rho \text{ A} \begin{Bmatrix} l - \frac{3\,l}{2} + \frac{2l}{3} \\ \frac{-l}{2} + \frac{2\,l}{3} \\ \frac{4l}{2} - \frac{4l}{3} \end{Bmatrix}$$

$$= \rho \, A \begin{Bmatrix} 0.166 \, l \\ 0.166 \, l \\ 0.166 \, l \end{Bmatrix}$$

$$= \rho \, A \, l \begin{Bmatrix} 0.166 \\ 0.166 \\ 0.166 \end{Bmatrix}$$

$$\{F\} = \rho \, A \, l \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \rho \, A \, l \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix}$$