

2.1.2 NULL SPACES AND RANGES

Null space (or) Kernel

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation. Then the set of all vectors x in V such that $T(x) = 0_W$ is called the null space (or kernel) of T . It is denoted by $N(T)$.

$$(i.e) N(T) = \{x \in V: T(x) = 0_W\}$$

Note: 0_W is the zero element of W .

Range or Image

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation. Then the subset of W consisting of all images under T of vectors in V is called range or image of T . It is denoted by $R(T)$.

$$(i.e) R(T) = \{T(x): x \in V\}$$

Theorem 2.1: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. Then

- (a) $N(T)$ is a subspace of V and
- (b) $R(T)$ is a subspace of W .

Proof: Given that V and W are vector spaces.

$T: V \rightarrow W$ is linear.

(a) To prove $N(T)$ is a subspace of V .

We have to prove for $\alpha, \beta \in F$ and $x, y \in N(T) \Rightarrow \alpha x + \beta y \in N(T)$

Since T is linear, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$T(0_V) = 0_W$; 0_V -zero vector of V and 0_W -zero vector of W .

$$\therefore 0_V \in N(T)$$

$\therefore N(T)$ is non-empty.

Let $x, y \in N(T)$ and $\alpha, \beta \in F$

$$\Rightarrow T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha 0_W + \beta 0_W = 0_W$$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) (\because T \text{ is linear})$$

$$= \alpha(0_W) + \beta(0_W) = 0_W$$

$$\therefore T(\alpha x + \beta y) = 0$$

$$\Rightarrow \alpha x + \beta y \in N(T)$$

$\therefore N(T)$ is subspace of V

(b) To prove $R(T)$ is subspace of W .

We have to prove for $\alpha, \beta \in F$ and $x, y \in R(T) \Rightarrow \alpha x + \beta y \in R(T)$

Since $T(0_V) = 0_W$ ($\because T$ is linear)

$$\Rightarrow 0_W \in R(T)$$

$\therefore R(T)$ is non-empty.

Let $x, y \in R(T)$ and $\alpha, \beta \in F$

Then there exists u and v in V such that

$$T(u) = x \text{ and } T(v) = y$$

$$\alpha x + \beta y = \alpha T(u) + \beta T(v)$$

$$= T(\alpha u + \beta v) \in R(T) [\because \alpha u + \beta v \in V]$$

$$\therefore \alpha x + \beta y \in R(T)$$

$\therefore R(T)$ is a subspace of W .

Theorem 2.2: Let V and W be vector spaces over a field F . Let $T: V \rightarrow W$ linear transformation which is onto. Then T maps a basis of V onto a basis of W .

Proof Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V .

We shall prove that $T(v_1), T(v_2), \dots, T(v_n)$ are linearly independent and that they span W .

$$\text{Now, } \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n) = 0$$

$$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_n v_n) = 0. [\because T \text{ is linear}]$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = 0$$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ (since v_1, v_2, \dots, v_n are linearly independent).

$\therefore T(v_1), T(v_2), \dots, T(v_n)$ are linearly independent.

Now, let $w \in W$. then since T is onto, there exists a vector $v \in V$ such $T(v) = w$

Since $L(S) = V$,

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Then

$$\begin{aligned} w &= T(v) \\ &= T(\alpha_1 v_1 + \dots + \alpha_n v_n) \\ &= \alpha_1 T(v_1) + \dots + \alpha_n T(v_n) [\because T \text{ is linear}] \end{aligned}$$

Thus w is a linear combination of the vectors $T(v_1), T(v_2), \dots, T(v_n)$. $\therefore T(v_1), T(v_2), \dots, T(v_n)$ span W and hence is a basis for W .

2.1.3. NULLITY AND RANK

Definition

Let V and W be vector spaces, and let $T: V \rightarrow W$ be a linear transformation. If $N(T)$ and $R(T)$ are finite-dimensional, then we define

$$\text{nullity}(T) = \dim[N(T)]$$

$$\text{rank}(T) = \dim[R(T)]$$

Note: If $\text{nullity}(T) = \{0\}$, then $\dim[N(T)] = 0$ i.e., $\text{nullity}(T) = 0$

Theorem 2.5: Rank-Nullity Theorem (or dimensional theorem) Let $T: V \rightarrow W$

be a linear transformation and V be a finite dimensional vector space. Then

$$\dim[R(T)] + \dim[N(T)] = \dim(V)$$

$$(i.e) \text{ran } k(T) + \text{nullity } (T) = \dim(V)$$

Proof:

Let V be a vector space of dimension m . i.e., $\dim(V) = m$. Since $N(T)$ is subspace of the finite dimensional vector space V dimension of $N(T)$ is also finite

$$\text{Let } \dim(N(T)) = n$$

Since $N(T)$ is a subspace of V , $n \leq m$

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis $N(T)$. Since $v_i \in N(T)$, for $1 \leq i \leq n$,

Then $T(v_i) = 0, 1 \leq i \leq n$

Since β is a basis of $N(T)$, β is linearly independent in $N(T)$.

Therefore β is linearly independent in V .

We shall extend this set β to a basis of the vector space V .

Let this basis of V be $\beta_1 = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_s\}$, where $n + s = m$.

Let $\gamma = \{T(u_1), T(u_2), \dots, T(u_s)\}$.

We shall show that this set γ is a basis of $R(T)$.

ie, to prove $L(\gamma) = R(T)$

and γ is linearly independent.

Since β_1 is a basis of V , it spans V . Hence the set

$\{T(v_1), T(v_2), \dots, T(v_n), T(u_1), T(u_2), \dots, T(u_s)\}$ spans $R(T)$

Since $T(v_i) = 0$, for $1 \leq i \leq n$

the set $\{T(u_1), T(u_2), \dots, T(u_s)\}$ spans $R(T)$.

$$L(\gamma) = R(T)$$

To prove is linearly independent.

$$\text{Let } a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0.$$

$$\Rightarrow T(a_1u_1 + a_2u_2 + \cdots + a_su_s) = 0 [\because T \text{ is linear}]$$

$$\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_su_s \in N(T)$$

Since $\beta = (v_1, v_2, \dots, v_n)$ is a basis in $N(T)$,

$$a_1u_1 + a_2u_2 + \cdots + a_su_s = b_1v_1 + b_2v_2 + \cdots + b_nv_n$$

$$\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_su_s - b_1v_1 - b_2v_2 - \cdots - b_nv_n = 0$$

Since β_1 is a basis of V , $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_s$ are linearly independent,

$$\therefore a_1 = a_2 = \cdots = a_s = 0, b_1 = b_2 = \cdots = b_n = 0$$

$$\therefore a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_s = 0$$

$\gamma = \{T(u_1), T(u_2), \dots, T(u_s)\}$ is linearly independent

$\therefore \gamma$ is a basis of $R(T)$

$$\therefore \dim R(T) = s$$

We have $m = n + s$

$$\therefore \dim(V) = \dim[N(T)] + \dim[R(T)]$$

$$\text{i.e., } \text{ran } k(T) + \text{nullity}(T) = \dim(V)$$

Theorem 2.6: Let V and W be vector space, and let $T: V \rightarrow W$ be transformation. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof:

Assume: T is 1 – 1 (one-to-one)

Let $u \in N(T)$. Then

$$T(u) = 0 = T(0)$$

$$\therefore T(u) = T(0)$$

$$\Rightarrow u = 0 (\because T \text{ is } 1 - 1)$$

$$\therefore N(T) = \{0\}$$

Conversely, assume that $N(T) = \{0\}$

$$\text{Let } T(u) = T(v).$$

$$T(u) - T(v) = 0$$

$$T(u - v) = 0 (\because T \text{ is linear})$$

$$\Rightarrow u - v \in N(T) = \{0\}$$

$$\therefore u - v = 0$$

$$\therefore u = v$$

$\therefore T$ is 1 - 1 (one-to-one).

Theorem 2.7: If V and W be finite dimensional over F and $T: V \rightarrow W$ be Then the following are equivalent.

- 1 T is one-to-one
- 2 T is onto
- 3 $\text{rank}(T) = \dim(W)$

Proof:

By dimensional theorem we have

$$\text{rank}(T) + \text{nullity}(T) = \dim(V) \dots ($$

$$T \text{ is one-to-one} \Leftrightarrow N(T) = \{0\}$$

$$\Leftrightarrow \text{nullity}(T) = 0$$

$$\Leftrightarrow \text{rank}(T) = \dim V \text{ [usin } g(1)]$$

$$\Leftrightarrow \dim R(T) = \dim W \text{ [} T \text{ is } 1 - 1]$$

$$\Leftrightarrow R(T) = W [\because R(T) \subseteq W \text{ with same rank}]$$

$$\Leftrightarrow T' \text{ is onto.}$$

2. 1.4. PROBLEMS UNDER RANK AND NULLITY

Let V and W be vector space and let $T: V \rightarrow W$ be linear map

- $N(T) = \{x \in V: T(x) = 0_W\}$
- $\text{nullity}(T) = \dim(N(T))$
- $R(T) = \{T(x): x \in V\}$
- $\text{rank}(T) = \text{image}(R(T))$

Example 17. Let $T: R^3 \rightarrow R^2$ by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find N and $R(T)$

Sol: To find $N(T)$:

$$N(T) = \{(a_1, a_2, a_3) \in R^3: T(a_1, a_2, a_3) = 0\}$$

$$\text{Let } T(a_1, a_2, a_3) = 0$$

$$(a_1 - a_2, 2a_3) = 0$$

Equating each terms to zero, we get

$$2a_3 = 0$$

$$a_3 = 0$$

$$a_1 - a_2 = 0$$

$$a_1 = a_2$$

$$N(T) = \{(a_1, a_2, a_3)\}$$

$$= \{(a_1, a_1, 0): a_1 \in R\}$$

To find $R(T)$:

The usual basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$$T(1,0,0) = (1,0)$$

$$T(0,1,0) = (-1,0) = -1(1,0)$$

$$T(0,0,1) = (0,2) = 2(0,1)$$

$$\begin{aligned} \text{Image}(T) &= \text{span}\{(1,0), -(1,0), 2(1,0)\} \\ &= \text{span}\{(1,0), (1,0)\} \begin{cases} * -(1,0) \text{ is depending on } (1,0) \\ 2(1,0) \text{ is multiple of } (1,0) \end{cases} \\ &= \{x(1,0) + y((1,0))\} \\ &= \{(x, y)\} \\ &= R^2 \end{aligned}$$

Example 18. Let $T: R^2 \rightarrow R^3$ be a linear map defined by $T(a_1, a_2) = (a_1 - a_2, 0, 0)$. Find nullity (T) and rank(T).

Sol: To find nullity (T) :

$$N(T) = \{(a_1, a_2) \in R^2: T(a_1, a_2) = 0\}$$

$$\text{Let } T(a_1, a_2) = 0$$

$$(a_1 - a_2, 0, 0) = (0, 0, 0)$$

$$a_1 - a_2 = 0$$

$$a_1 = a_2$$

$$N(T) = \{(a_1, a_1) / a_1 \in R\}$$

$$= \{(1,1)a_1 / a_1 \in R\}$$

The basis of $N(T)$ is $\beta = \{(1,1)\}$

The nullity of $T = \dim[N(T)] = 1$

To find range (T) :

The usual basis of R^2 is $\beta = \{(1,0), (0,1)\}$

Given, $T(a_1, a_2) = (a_1 - a_2, 0, 0)$.

$$T(1,0) = (1,0,0)$$

$$T(0,1) = (-1,0,0)$$

The image of usual basis span Image (T)

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \rightarrow R_2 + R_1$$

The basis of $R(T)$ is the non-zero row of the echelon matrix.

Therefore the basis of $R(T)$ is $\gamma = \{(1,0,0)\}$.

$$\text{rank}(T) = \dim[R(T)] = 1$$

Example (19) Let $T: R^2 \rightarrow R^3$ be the linear mapping defined by $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$

Find the basis and dimension of (a) null space of T (b) Range of T

Sol: (a) To find (null space) kernel of T :

$$\text{Let } T(a_1, a_2) = 0$$

$$(a_1 + a_2, a_1 - a_2, a_2) = (0,0,0)$$

Equating the like terms

$$a_1 + a_2 = 0 \dots (1)$$

$$a_1 - a_2 = 0 \dots (2)$$

$$a_2 = 0$$

$$(1) \Rightarrow a_2 = 0$$

$$\text{kernel of } T = N(T) = \{(0,0)\}$$

$$\text{The nullity of } T = \dim(N(T)) = 0$$

(b) To find range of T :

$$\text{The usual basis of } R^2 \text{ is } \beta = \{(1,0), (0,1)\}$$

$$\text{Given, } T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$$

$$T(1,0) = (1,1,0)$$

$$T(0,1) = (1,-1,0)$$

The image of usual basis span $\text{Image}(T)$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} R_2 \rightarrow R_2 - R_1$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $y = \{(1,0,1), (0,-2,0)\}$ forms a basis for $\text{Im}(T)$.

$$\text{Hence } \dim[\text{Im}(T)] = 2$$

$$\text{i.e., Rank}(T) = 2$$

Example 20. Let $T: R^3 \rightarrow R^3$ be the linear mapping defined by $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

Find the basis and dimension of (a) Kernel (b) Image of T

Sol: (a) To find kernel of T :

$$\text{Let } T(a_1, a_2, a_3) = 0$$

$$(a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3) = (0,0,0)$$

$$a_1 + 2a_2 - a_3 = 0 \dots (1)$$

$$a_2 + a_3 = 0 \dots (2)$$

$$a_1 + a_2 - 2a_3 = 0 \dots (3)$$

Solve (1), (2)&(3)

The matrix of the given equations is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} R_3 \rightarrow R_3 + R_2$$

$$a_1 + 2a_2 - a_3 = 0 \dots (4)$$

$$a_2 + a_3 = 0 \dots (5)$$

Adding (4) and (5), we get

$$a_1 + 3a_2 = 0$$

$$a_1 = -3a_2 \dots (6)$$

a_1 is depending on a_2 .

Therefore the basis of $N(T)$ contains one element

$$(6) \Rightarrow \frac{a_1}{-3} = \frac{a_2}{1}$$

$$a_1 = -3, a_2 = 1$$

$$(5) \Rightarrow a_3 = -1$$

Basis of kernel of T is $\beta = \{(-3, 1, -1)\}$

$$\text{nullity}(T) = \dim(N(T)) = 1$$

(b) To find Image (T) :

The basis of R^3 is $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

$$T(1, 0, 0) = (1, 0, 1)$$

$$T(0, 1, 0) = (2, 1, 1)$$

$$T(0, 0, 1) = (-1, 1, -2)$$

The image of usual basis span Image (T)

Let
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 - R_2.$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $\gamma = \{(1, 0, 1), (0, 1, -1)\}$ form a basis for $\text{Im}(T)$.

$$\text{Hence } \dim[\text{Im}(T)] = 2$$

$$\text{i.e., Rank}(T) = 2$$

Example 21. Let $T: P_3(R) \rightarrow P_2(R)$ defined by $T[f(x)] = f'(x)$. find the nullity and rank of T .

Sol: Let $f(x) \in P_3(x)$. Then

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots (1)$$

$$f'(x) = a_1 + a_2x + a_3x^2$$

To find nullity (T) :

$$N(T) = \{f(x) \in P_3(R) : T(f(x)) = 0\} \dots (2)$$

Let $T(f(x)) = 0$. Then

$$f'(x) = 0$$

$$a_1 + a_2x + a_3x^2 = 0$$

$$a_1 + a_2x + a_3x^2 = 0 + 0x + 0x^2$$

$$a_1 = 0, a_2 = 0, a_3 = 0$$

Example 23. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem $T: V_2(R) \rightarrow V_2(R)$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

Sol: To find (T) :

$$\text{Let } T(x_1, x_2) = 0$$

$$(x_1 + x_2, x_1) = 0$$

$$x_1 + x_2 = 0 \dots (1)$$

$$x_1 = 0$$

$$(1) \Rightarrow x_2 = 0$$

$\therefore N(T)$ contain only zero element of $V_2(R)$.

$$\therefore N(T) = \{(0,0)\}$$

i.e the null space = $\{(0,0)\}$

$$\dim[N(T)] = 0$$

i.e nullity = 0

To find $R(T)$:

The standard basis $e_1 = (1,0)$, $e_2 = (0,1)$ of $V_2(R)$

$$\begin{aligned} T(e_1) &= T(1,0) \\ &= (1 + 0, 1) \end{aligned}$$

$$\begin{aligned} T(e_2) &= T(0,1) \\ &= (0 + 1, 0) \\ &= (1, 0) \end{aligned}$$

The image of usual basis span Image (T)

$$\begin{aligned} \text{Let } A &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} R_2 \rightarrow R_2 - R_1 \end{aligned}$$

This is in the echelon form there are two non-zero rows

Basis of Image (T) is $\gamma = \{(1,1), (0, -1)\}$

Therefore, rank of $T = 2$

Hence $R(T)$ is the subspace generated by $(1,1)$ and $(0, -1)$

$$\begin{aligned} R(T) &= x_1(1,1) + x_2(0, -1) \\ &= (x_1, x_1) + (0, -x_2) \\ &= (x_1, x_1 - x_2) \text{ for all } x_1, x_2 \in R \end{aligned}$$

i.e the range space = $\{x_1, x_1 - x_2\} = V_2(R)$

Rank + nullity = 2 + 0

$$= 2$$

$$= \dim[V_2(R)]$$

Hence the nullity theorem is verified.

Example(24) Let $T: R^3 \rightarrow R^3$ defined by $Y(x,y,z)=(x+y,x-y,2x+z)$. Find the range space, null-space, rank and nullity of T and verify rank+nullity of T = $\dim(R^3)$.

Sol: To find (T) :

$$T(x, y, z) = 0(x, x - 2y, 2x, 2x - 2y + z) = (0,0,0)$$

$$x_1 + x_2 = 0 \dots (1)$$

$$x_1 - x_2 = 0 \dots (2)$$

$$2x_1 + x_3 = 0 \dots (3)$$

$$(1) + (2) \Rightarrow 2x_1 = 0$$

$$x_1 = 0$$

$$(1) \Rightarrow x_2 = 0$$

$$(3) \Rightarrow x_3 = 0$$

$$\therefore N(T) = \{(0,0,0)\}$$

$$= \{0\}$$

$$\Rightarrow \dim[N(T)] = 0$$

$$\Rightarrow \text{nullity} = 0$$

To find $R(T)$:

The standard basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

$$T(x, y, z) = (x + y, x - y, 2x + z)$$

$$T(1,0,0) = (1 + 0, 1 - 0, 0 + 0)$$

$$= (1,1,0)$$

$$T(0,0,0) = (0 + 1, 0 - 1, 0 + 0)$$

$$= (1, -1, 0)$$

$$T(0,0,1) = (0 + 0, 0 - 0, 0 + 1)$$

$$= (0, 0, 1)$$

The image of usual basis span Image (T)

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This in echelon form and there are three non-zero rows.

$$\dim[R(T)] = 3$$

$$\text{i.e rank of } T = 3$$

$$R(T) = \text{the subspace generated by } (1,1,0), (0, -2, 0), (0,0,1)$$

$$= x(1,1,2) + y(0, -2, -2) + z(0,0,1)$$

$$= (x, x, 2x) + (0, -2y, -2y) + (0,0, z)$$

$$R(T) = (x, x - 2y, 2x, 2x - 2y + z)$$

$$\text{Rank} + \text{nullity} = 3 = \dim(R^3)$$

Example 25. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem defined by

$$T: V_3(R) \rightarrow V_2(R) \text{ by}$$

$$T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$$

Sol: To find the kernel,

Since the linear transformation is not given, first find the linear transformation

The usual basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$$

$$\text{Given } T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$$

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

$$= x_1e_1 + x_2e_2 + x_3e_3$$

$$T(x_1, x_2, x_3) = T(x_1e_1 + x_2e_2 + x_3e_3)$$

$$= x_1T(e_1) + x_2T(e_2) + x_3T(e_3)$$

$$= x_1(2,1) + x_2(0,1) + x_3(1,1)$$

$$= (2x_1, x_1) + (0, x_2) + (x_3, x_3)$$

$$= (2x_1 + x_3, x_1 + x_2 + x_3)$$

$$N(T) = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = 0\}$$

$$\text{Put } T(x_1, x_2, x_3) = 0$$

$$(2x_1 + x_3, x_1 + x_2 + x_3) = (0,0)$$

$$2x_1 + x_3 = 0 \dots (1)$$

$$x_1 + x_2 + x_3 = 0$$

$$(1) \Rightarrow x_3 = -2x_1$$

$$(2) \Rightarrow x_1 + x_2 - 2x_1 = 0$$

$$x_2 - x_1 = 0$$

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$$(3) \Rightarrow x_3 = -2$$

The basis of $N(T)$ is $\beta = \{(1,1,-2)\}$

$N(T)$ = the subspace generated by $(1,1,-2)$

$$= \{(1,1,-2)x_1\}$$

$$= \{(x_1, x_1, -2x_1)\}$$

$$\therefore \dim[N(T)] = 1$$

$$\text{i.e nullity } (T) = 1$$

To find $\dim[R(T)]$

$$\text{Given } T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$$

The image of usual basis span Image (T) .

$$\begin{aligned} \text{Let } A &= \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} R_1 \leftrightarrow R_3 \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} R_3 \rightarrow R_3 - 2R_1 \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} R_3 \rightarrow R_3 + R_2 \end{aligned}$$

This is the echelon form and there are 2 non-zero rows in it.

The basis of $R(T)$ is $\gamma = \{(1,1), (0,1)\}$

$$\therefore \dim[R(T)] = 2$$

$$\text{i.e rank of } T = 2$$

Range space = the subspace generated by $(1,1)$ and $(0,1)$

$$= x_1(1,1) + x_2(0,1)$$

$$= (x_1, x_1) + (0, x_2)$$

$$= (x_1, x_1 + x_2)$$

$$\therefore \text{Range space} = \{(x_1, x_1 + x_2) \mid x_1, x_2 \in R\}$$

$$\text{Rank}(T) + \text{nullity}(T) = 2 + 1$$

$$= 3$$

$$= \dim(R^3)$$

