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DEPARTMENT OF MATHEMATICS

**NAME OF THE SUBJECT : TRANSFORMS & PARTIAL
DIFFERENTIAL
EQUATION**

SUBJECT CODE : MA8353

REGULATION : 2017

UNIT – I : PARTIAL DIFFERENTIAL EQUATION

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

Notations: If $z = f(x, y)$ then

$$p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}; r = \frac{\partial^2 z}{\partial x^2}; s = \frac{\partial^2 z}{\partial x \partial y}; t = \frac{\partial^2 z}{\partial y^2}$$

Formation of PDE by eliminating arbitrary constants:

Let the given equation be $z = f(x, y, a, b)$ ----- (1)

Step 1: Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = p = f'(x, y, a, b) \text{ ----- (2)}$$

Step 2: Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = q = f'(x, y, a, b) \text{ ----- (3)}$$

Step 3: Eliminate a & b from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from**

$$z = (x - a)^2 + (y - b)^2$$

Solution:

$$\text{Given } z = (x - a)^2 + (y - b)^2 \text{ ----- (1)}$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = 2(x - a) + 0$$

$$p = 2(x - a) \text{ ----- (2)}$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 2(y - b)$$

$$q = 2(y - b) \text{ ----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow (x - a) = \frac{p}{2} \text{ ----- (4)}$$

$$(3) \Rightarrow y - b = \frac{q}{2} \text{ ----- (5)}$$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

The required the PDE is

$$\boxed{p^2 + q^2 = 4z}$$

2. **Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from**

$$z = (x^2 + a)(y^2 + b).$$

Solution:

$$\text{Given } z = (x^2 + a)(y^2 + b) \text{ ----- (1)}$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \text{ ----- (2)}$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \text{ ----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow (y^2 + b) = \frac{p}{2x} \text{-----(4)}$$

$$(3) \Rightarrow x^2 + b = \frac{q}{2y} \text{-----(5)}$$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

The required the PDE is

$$\boxed{4xyz = pq}$$

3. **Find the PDE of all planes having equal intercepts on the x and y axis.**

Solution:

The intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given that equal intercepts on the x & y axis $\Rightarrow a = b$

$$\therefore \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \text{-----(1)}$$

Diff Partially w.r.t x

$$\frac{1}{a} + 0 + \frac{1}{c} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} p \text{-----(2)}$$

Diff Partially w.r.t y

$$0 + \frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} q \text{-----(3)}$$

From (2) & (3) $\frac{-1}{c} p = \frac{-1}{c} q$ The required the PDE is $\boxed{p = q}$

4. **Obtain the partial differential equation by eliminating arbitrary constants 'a' and 'b' from**

$$(x-a)^2 + (y-b)^2 + z^2 = r^2$$

Solution:

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \text{-----(1) - (1)}$$

Diff Partially w.r.t x

$$2(x-a)(1-0) + 0 + 2z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow 2(x-a) + 2zp = 0 \text{----- (2)}$$

Diff Partially w.r.t y

$$0 + 2(y-b)(1-0) + 2z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 2(y-b) + 2zq = 0 \text{----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow x-a = -zp \text{----- (4)}$$

$$(3) \Rightarrow y-b = -zq \text{----- (5)}$$

Sub (4) & (5) in (1)

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

The required PDE is $\boxed{z^2(p^2 + q^2 + 1) = 1}$

Formation of PDE by eliminating arbitrary functions:

1. **Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ and form the PDE.**

Solution:

$$z = f\left(\frac{y}{x}\right) \text{-----(1)}$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right) \times \left(\frac{-y}{x^2}\right) \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \text{-----(2)}$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right) \text{-----(3)}$$

From (1) & (2) $\frac{p}{q} = \frac{-y}{x} \Rightarrow \boxed{px + qy = 0}$

2. **Form the partial differential equation by eliminating f from $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.**

Solution:

Given $z = x^2 + 2f\left(\frac{1}{y} + \log x\right) \text{-----(1)}$

Differentiate (1) partially w.r. t x

$$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(0 + \frac{1}{x}\right)$$

$$p = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = (p - 2x)\frac{x}{2} \text{-----(2)}$$

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2} + 0\right)$$

$$q = \frac{-2}{y^2} f'\left(\frac{1}{y} + \log x\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = \frac{-qy^2}{2} \text{-----(3)}$$

Eliminating f' from (2) & (3)

$$(p - 2x)\frac{x}{2} = \frac{-qy^2}{2} \Rightarrow (px - 2x^2) = -qy^2$$

$$\Rightarrow \boxed{px + qy^2 = 2x^2}$$

Formation of PDE by eliminating f from $f(u, v) = 0$ -----(1)

Method 1:

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

Method 2:

The required PDE is $Pp + Qq = R$

Where

$$P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}; Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}; R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$$

1. **Form the PDE from $\phi(ax + by + cz, x^2 + y^2 + z^2) = 0$**

Solution:

Given $\phi(ax + by + cz, x^2 + y^2 + z^2) = 0$

This is of the form $f(u, v) = 0$ where $u = ax + by + cz$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$$\Rightarrow p(2bz - 2cy) - q(2az - 2cx) + 1(2az - 2cx) = 0$$

$$\div 2 \Rightarrow \boxed{(bz - cy)p + (cx - az)q + (az - cx) = 0}$$

2. **Form the PDE from $\phi(x^2 + y^2 + z^2, xyz) = 0$**

Solution:

Given $\phi(x^2 + y^2 + z^2, xyz) = 0$

This is of the form $f(u, v) = 0$ where $u = x^2 + y^2 + z^2$ & $v = xyz$

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 0$$

$$\Rightarrow p(2xy^2 - 2xz^2) - q(2x^2y - 2yz^2) + 1(2x^2z - 2y^2z) = 0$$

$$\div 2 \Rightarrow \boxed{x(y^2 - z^2)p + y(z^2 - x^2) + z(x^2 - y^2) = 0}$$

3. **Form the PDE from $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$**

Solution:

Given $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$

This is of the form $f(u, v) = 0$ where $u = \frac{y}{x}$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$$\Rightarrow p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$$

$$\Rightarrow \frac{2zp}{x} + \frac{2yzq}{x^2} - 2\left(\frac{y^2}{x^2} + 1\right) = 0$$

$$\div 2 \Rightarrow \frac{zp}{x} + \frac{yzq}{x^2} - \left(\frac{y^2 + x^2}{x^2}\right) = 0$$

$$\Rightarrow \frac{xzp + yzq - (y^2 + x^2)}{x^2} = 0$$

$$xzp + yzq - (y^2 + x^2) = 0$$

Solutions of standard types of First order PDE's:

Different solutions of PDE:

Complete Integral (or) Complete Solution:

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

Singular Integral (or) Singular Solution:

Consider a PDE of first order as $f(x, y, z, p, q) = 0$ ----- (1)

It's complete integral may be, $f(x, y, z, a, b) = 0$ ----- (2)

Diff (2) partially with respect to a & b respectively,

$$\frac{\partial g}{\partial a} = 0 \text{ ----- (3)}$$

$$\frac{\partial g}{\partial b} = 0 \text{ ----- (4)}$$

Eliminating a & b from (3) & (4) will get the Singular integral.

General Integral (or) Complete solution:

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.

(or) A solution which contains the maximum possible number of arbitrary functions.

Type I:

Equations of the form $f(p, q) = 0$ ----- (1)

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c$ ----- (2)

Let $p = a$ & $q = b$ in (1)

$f(a, b) = 0$ and represent $b = \phi(a)$

$\therefore (1) \Rightarrow z = ax + \phi(a)y + c$ ----- (3)

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = g(a)$ in (3)

(3) $\Rightarrow z = ax + \phi(a)y + g(a)$ ----- (4)

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)y + g'(a)$ ----- (5)

Eliminating a from (4) & (5) we get general integral.

1. Find the complete integral of $p + q = pq$

Solution:

Given $p + q = pq$ ----- (1)

This of the form $f(p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c$ ----- (2)

Let $p = a$ & $q = b$ in (1)

$$(1) \Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \frac{a}{a-1}$$

Sub b in (2)

$$z = ax + \left(\frac{a}{a-1} \right) y + c$$

This is the required complete integral.

2. **Find the complete integral of $p + q = 1$**
Solution:
 Given $p + q = 1$ -----(1)
 This of the form $f(p, q) = 0$
To find Complete Integral:
 Let the complete solution of (1) is $z = ax + by + c$ -----(2)
 Let $p = a$ & $q = b$ in (1)
 $(1) \Rightarrow a + b = 1 \Rightarrow b = 1 - a$
 Sub b in (2)

$$z = ax + (1 - a)y + c$$

 This is the required complete integral.

3. **Solve $\sqrt{p} + \sqrt{q} = 1$**
Solution:
 Given $\sqrt{p} + \sqrt{q} = 1$ -----(1)
 This of the form $f(p, q) = 0$
To find Complete Integral:
 Let the complete solution of (1) is $z = ax + by + c$ -----(2)
 Let $p = a$ & $q = b$ in (1)
 $(1) \Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a} \Rightarrow b = (1 - \sqrt{a})^2$
 Sub b in (2)

$$z = ax + (1 - \sqrt{a})^2 y + c$$
 -----(3)
 This is the required complete integral.
To Find Singular Integral:
 Diff (3) partially with respect to c
 $0 = 1$ which is impossible
 There is no singular integral for this type.
To find General integral:
 Put $c = f(a)$ in (3)
 $(3) \Rightarrow z = ax + (1 - \sqrt{a})^2 y + f(a)$ -----(4)
 Diff. (4) partially with respect to a
 $0 = x(1) + 2(1 - \sqrt{a}) \frac{1}{2\sqrt{a}} y + f'(a)$ -----(5)
 Eliminate a from (4) & (5) we get the general integral.

Type II:
 Equations of the form $z = px + qy + f(p, q)$ -----(1)
 To find Complete Integral:
 Put $p = a$ & $q = b$ in (1)
 $\therefore (1) \Rightarrow z = ax + by + f(a, b)$ -----(2)
To Find Singular Integral:
 Diff (2) partially with respect to a
 $0 = x(1) + 0 + f'(a, b)$ -----(3)
 Diff (2) partially with respect to b
 $0 = 0 + y(1) + f'(a, b)$ -----(4)
 Eliminating a & b from (2) using (3) & (4), we get the singular integral.
To find General integral:
 Put $b = \phi(a)$ in (2)
 $(2) \Rightarrow z = ax + \phi(a)y + g(a)$ -----(5)
 Diff (4) partially with respect to a

$$0 = x(1) + \phi'(a)y + g'(a) \text{ -----(6)}$$

Eliminating a from (5) & (6) we get General integral.

1. **Solve** $z = px + qy + p^2q^2$

Solution:

Given $z = px + qy + p^2q^2$ -----(1)

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2b^2}$$
 -----(2)

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + 2ab^2 \Rightarrow x + 2ab^2 = 0 \Rightarrow x = -2ab^2$$
 -----(3)

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 + 2a^2b \Rightarrow y + 2a^2b = 0 \Rightarrow y = -2a^2b$$
 -----(4)

Eliminating a & b from (2) using (3) & (4)

$$(3) \Rightarrow \frac{x}{b} = -2ab$$
 -----(5)

$$(4) \Rightarrow \frac{y}{a} = -2ab$$
 -----(6)

From (5) & (6)

$$\frac{x}{b} = \frac{y}{a} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{b} = k \text{ \& \ } \frac{y}{a} = k$$

$$\Rightarrow b = \frac{x}{k} \text{ \& \ } a = \frac{y}{k}$$
 -----(7)

Sub a & b in (2)

$$(2) \Rightarrow z = \frac{y}{k}x + \frac{x}{k}y + \frac{y^2}{k^2} \frac{x^2}{k^2}$$

$$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4}$$

$$z = \frac{2xy}{k} + \frac{x^2y^2}{k^4}$$
 -----(8)

To find k

Sub (7) in (3) (or) (4)

$$(3) \Rightarrow x = -2 \frac{y}{k} \frac{x^2}{k^2} \Rightarrow x = \frac{-2x^2y}{k^3} \Rightarrow k^3 = -2xy$$

$$(8) \Rightarrow z = \frac{2xy}{k} + \frac{x^2y^2}{k(-2xy)} \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$$

$$z = \frac{4xy - xy}{2k} \Rightarrow z = \frac{3xy}{2k} \Rightarrow z^3 = \frac{27x^3y^3}{8k^3} \Rightarrow z^3 = \frac{27x^3y^3}{8(-2xy)}$$

$$\boxed{16z^3 = -27x^2y^2}$$

This is the required singular integral.

To find General integral:

Put $b = \phi(a)$ in (2)

$$(2) \Rightarrow z = ax + f(a)y + a^2[f(a)]^2$$
 -----(9)

Diff (9) partially with respect to a
 $0 = x(1) + f'(a)y + 2f(a)f'(a) \text{ -----(10)}$
 Eliminating a from (9) & (10) we get General integral.

2. **Find the singular integral of $z = px + qy + p^2 + pq + q^2$**

Solution:

Given $z = px + qy + p^2 + pq + q^2 \text{ -----(1)}$

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2 + ab + b^2} \text{ -----(2)}$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + 2a + b + 0 \Rightarrow 2a + b = -x \text{ -----(3)}$

Diff (2) partially with respect to b

$0 = 0 + y(1) + 0 + a + 2b \Rightarrow a + 2b = -y \text{ -----(4)}$

Eliminating a & b from (2) using (3) & (4)

$(4) \times 2 \Rightarrow 2a + 4b = -2y \text{ -----(5)}$

$(3) - (5) \Rightarrow -3b = -x + 2y \Rightarrow \boxed{b = \frac{x - 2y}{3}}$

Sub the value of b in (3)

$2a + b = -x \Rightarrow 2a = -x - b \Rightarrow 2a = -x - \left(\frac{x - 2y}{3}\right)$

$2a = \frac{-3x - x + 2y}{3} \Rightarrow 6a = -4x + 2y \Rightarrow \boxed{a = \frac{y - 2x}{3}}$

Sub the value of a & b in (2)

$$z = \left(\frac{y - 2x}{3}\right)x + \left(\frac{x - 2y}{3}\right)y + \left(\frac{y - 2x}{3}\right)^2 + \left(\frac{y - 2x}{3}\right)\left(\frac{x - 2y}{3}\right) + \left(\frac{x - 2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 - 4xy + 4y^2}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 - 4xy + 4y^2}{9}$$

$$z = \frac{3xy - 3x^2 + 3xy - 6y^2 + y^2 - 4xy + 4y^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2}{9}$$

$\boxed{9z = -4x^2 + y^2 + xy}$

3. **Solve $z = px + qy + \sqrt{p^2 + q^2 + 1}$**

Solution:

Given

$z = px + qy + \sqrt{p^2 + q^2 + 1} \text{ -----(1)}$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$(1) \Rightarrow \boxed{z = ax + by + \sqrt{a^2 + b^2 + 1}} \text{ -----(2)}$

This is required complete integral.

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2a) \Rightarrow \boxed{x = \frac{-a}{\sqrt{a^2 + b^2 + 1}}} \text{ -----(3)}$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2b) \Rightarrow \boxed{y = \frac{-b}{\sqrt{a^2 + b^2 + 1}}} \text{-----(4)}$$

Eliminating a & b from (2) using (3) & (4)

$$(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \left(\frac{-a}{\sqrt{a^2 + b^2 + 1}}\right)^2 + \left(\frac{-b}{\sqrt{a^2 + b^2 + 1}}\right)^2$$

$$x^2 + y^2 = \frac{a^2}{a^2 + b^2 + 1} + \frac{b^2}{a^2 + b^2 + 1}$$

$$x^2 + y^2 = \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{a^2 + b^2 + 1 - a^2 - b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{1}{a^2 + b^2 + 1}$$

$$\Rightarrow 1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Taking square root on both sides

$$\Rightarrow \boxed{\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}} \text{-----(5)}$$

Sub (5) in (2) and (3)

$$(3) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1 - x^2 - y^2}}} \Rightarrow x = -a\sqrt{1 - x^2 - y^2} \Rightarrow \boxed{a = \frac{-x}{\sqrt{1 - x^2 - y^2}}}$$

$$(4) \Rightarrow y = \frac{-b}{\frac{1}{\sqrt{1 - x^2 - y^2}}} \Rightarrow y = -b\sqrt{1 - x^2 - y^2} \Rightarrow \boxed{b = \frac{-y}{\sqrt{1 - x^2 - y^2}}}$$

Sub (5), a & b in (2)

$$(2) \Rightarrow z = \left(\frac{-x}{\sqrt{1 - x^2 - y^2}}\right)x + \left(\frac{-y}{\sqrt{1 - x^2 - y^2}}\right)y + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \frac{-x^2}{\sqrt{1 - x^2 - y^2}} + \frac{-y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2}$$

Squaring on both sides

$$z^2 = 1 - x^2 - y^2 \Rightarrow \boxed{x^2 + y^2 + z^2 = 1}$$

4. **Find the singular integral of** $z = px + qy + p^2 - q^2$

Solution:

Given $z = px + qy + p^2 - q^2$ ----- (1)

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow z = ax + by + a^2 - b^2 \quad \text{-----(2)}$$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} \quad \text{-----(3)}$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} \quad \text{-----(4)}$$

Sub a & b in (2)

$$\therefore (2) \Rightarrow z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow \boxed{4z = -x^2 + y^2}$$

This is the required singular integral.

Type III:

Equations of the form $f(z, p, q) = 0$ ----- (1)

In this type x & y do not appear explicitly.

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ ----- (2)

Let $x + ay = u$

$$(2) \Rightarrow z = f(u) \quad \text{-----(3)}$$

By total derivative,

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{du} \quad (1) \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a \frac{dz}{du} \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$$

Substitute the value of p & q in (1)

$$(1) \Rightarrow f\left(z, \frac{dz}{du}, a \frac{dz}{du}\right) = 0$$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

1. Solve $p(1 + q) = qz$.

Solution:

$$\text{Given } p(1 + q) = qz \quad \text{-----(1)}$$

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ ----- (2)

Let $x + ay = u \Rightarrow z = f(u)$

$$\text{Then } p = \frac{dz}{du} \quad \& \quad q = a \frac{dz}{du}$$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du} \left(1 + a \frac{dz}{du}\right) = a \frac{dz}{du} z$$

$$1 + a \frac{dz}{du} = az$$

$$\frac{dz}{du} = az - 1$$

$$\frac{dz}{az - 1} = du$$

Integrating on both sides

$$\int \frac{dz}{az - 1} = \int du$$

$$u = \log(az - 1) + c \quad \because \int \frac{f'(x)}{f(x)} dx = \log f(x)$$

$$\boxed{x + ay = \log(az - 1) + c}$$

This is the required complete integral.

Other solutions can be obtained as usual.

2.

Solve $z^2 = 1 + p^2 + q^2$

Solution:

Given $z^2 = 1 + p^2 + q^2$ -----(1)

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ -----(2)

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a \frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow z^2 = \left(\frac{dz}{du}\right)^2 + \left(a \frac{dz}{du}\right)^2 + 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow (1 + a^2) \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{1 + a^2}$$

Taking square root on both sides

$$\frac{dz}{du} = \frac{\sqrt{z^2 - 1}}{\sqrt{1 + a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides

$$\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} u + c \quad \because \int \frac{dz}{\sqrt{x^2 - 1}} = \cosh^{-1} x$$

$$\boxed{\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} (x + ay) + c} \quad \because u = x + ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

3. Solve $p(1 - q^2) = q(1 - z)$

Solution:

Given $p(1 - q^2) = q(1 - z)$ -----(1)

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ -----(2)

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a \frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du} \left[1 - \left(a \frac{dz}{du} \right)^2 \right] = a \frac{dz}{du} (1 - z)$$

$$1 - a + az = a^2 \left(\frac{dz}{du} \right)^2$$

Taking square root on both sides

$$a \frac{dz}{du} = \sqrt{1 - a + az}$$

$$\frac{a dz}{\sqrt{1 - a + az}} = du$$

Integrating on both sides

$$a \left(\frac{2\sqrt{1 - a + az}}{a} \right) = u + c \quad \because \int \frac{1}{\sqrt{ax}} dx = \frac{1}{a} (2\sqrt{x})$$

$$\boxed{2\sqrt{1 - a + az} = x + ay + c} \quad \because u = x + ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

Type IV:

Equations of the form $f_1(x, p) = f_2(y, q)$ -----(1)

To find Complete Integral:

Let $f_1(x, p) = f_2(y, q) = a$ (say)

$\therefore f_1(x, p) = a$; $f_2(y, q) = a$

From the above we get $p = f_1(x, a)$; $q = f_2(y, b)$

Substitute the value of p & q in $z = \int p dx + \int q dy$

Integrating we get complete integral

Other solutions can obtain as usual.

1. Solve $p^2 + q^2 = x^2 + y^2$

Solution:

Given $p^2 + q^2 = x^2 + y^2$

$$p^2 - x^2 = y^2 - q^2$$
 -----(1)

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p^2 - x^2 = y^2 - q^2 = a^2$ (say)

$\therefore p^2 - x^2 = a^2$; $y^2 - q^2 = a^2$

$\therefore p^2 = a^2 + x^2$; $q^2 = y^2 - a^2$

$$\boxed{p = \sqrt{a^2 + x^2}} \quad ; \quad \boxed{q = \sqrt{y^2 - a^2}}$$

Substitute the value of p & q in

$$z = \int \sqrt{x^2 + a^2} dx + \int \sqrt{y^2 - a^2} dy$$

$$z = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a} \right) + c$$

$$1. \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right); \int \sqrt{y^2 - a^2} dy = \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a} \right)$$

Integrating we get complete integral

Other solutions can obtain as usual.

2. **Find the complete integral of** $p^2 y(1+x^2) = qx^2$

Solution:

Given $p^2 y(1+x^2) = qx^2$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} \text{ ----- (1)}$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a$ (say)

$$\frac{p^2(1+x^2)}{x^2} = a ; \frac{q}{y} = a$$

$$p^2 = a \frac{x^2}{1+x^2} ; q = ay$$

$$p = \frac{\sqrt{a} x}{\sqrt{1+x^2}} ; q = ay$$

Substitute the value of p & q in

$$z = \int \frac{\sqrt{a} x}{\sqrt{1+x^2}} dx + \int ay dy$$

$$\text{let } 1+x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$$

$$z = \sqrt{a} \int \frac{1}{\sqrt{t}} \frac{dt}{2} + a \frac{y^2}{2}$$

$$z = \frac{\sqrt{a}}{2} 2\sqrt{t} + \frac{ay^2}{2} \quad \because \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$$

$$z = \sqrt{a} \sqrt{1+x^2} + \frac{ay^2}{2} + c$$

3. **Find the complete integral of** $p + q = \sin x + \sin y$

Solution:

Given $p + q = \sin x + \sin y$

$$p - \sin x = \sin y - q \text{ ----- (1)}$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p - \sin x = \sin y - q = a$ (say)

$$\therefore p - \sin x = a ; \sin y - q = a$$

$$\therefore p = \sin x + a ; q = \sin y - a$$

Substitute the value of p & q in

$$z = \int (\sin x + a) dx + \int (\sin y - a) dy$$

$$z = \cos x + ax + \cos y - ay + c$$

$$z = \cos x + \cos y - a(x - y) + c$$

This is the required complete integral

Type V:

Equations reducible to standard form (Identification:

1.

Solve $z^2(p^2 + q^2) = x^2 + y^2$

Solution:

Given $z^2(p^2 + q^2) = x^2 + y^2$

This is of the form equations reducible to standard form

$$z^2 p^2 + z^2 q^2 = x^2 + y^2$$

$$(zp)^2 + (zq)^2 = x^2 + y^2$$

$$\left(\frac{z \partial z}{\partial x}\right)^2 + \left(\frac{z \partial z}{\partial y}\right)^2 = x^2 + y^2 \text{ ----- (1)} \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

Let $z \partial z = \partial Z$; $\partial x = \partial X$; $\partial y = \partial Y$

Integrating $\frac{z^2}{2} = Z$; $x = X$; $y = Y$

$$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = X^2 + Y^2$$

$$\boxed{P^2 + Q^2 = X^2 + Y^2} \quad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form $F_1(X, P) = F_2(Y, Q)$

To find Complete Integral:

Let $P^2 - X^2 = Y^2 - Q^2 = a^2$ (say)

$$\therefore P^2 - X^2 = a^2 ; Y^2 - Q^2 = a^2$$

$$\therefore P^2 = a^2 + X^2 ; Q^2 = Y^2 - a^2$$

$$\boxed{P = \sqrt{a^2 + X^2} ; Q = \sqrt{Y^2 - a^2}}$$

Substitute the value of p & q in

$$z = \int \sqrt{X^2 + a^2} dX + \int \sqrt{Y^2 - a^2} dY$$

$$Z = \frac{X}{2} \sqrt{X^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{X}{a}\right) + \frac{Y}{2} \sqrt{Y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{Y}{a}\right) + c$$

$$\frac{z^2}{2} = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a}\right) + \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a}\right) + c$$

$$\boxed{z^2 = x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \left(\frac{x}{a}\right) + y \sqrt{y^2 - a^2} + a^2 \cosh^{-1} \left(\frac{y}{a}\right) + 2c}$$

$$\because \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a}\right); \int \sqrt{y^2 - a^2} dy = \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a}\right)$$

This is the required complete integral

Other solutions can obtain as usual.

2.

Solve $p^2 + x^2 y^2 q^2 = x^2 z^2$

Solution:

Given $p^2 + x^2 y^2 q^2 = x^2 z^2$

This is of the form equations reducible to standard form

$$\div x \text{ on both sides } \frac{p^2}{x^2} + y^2 q^2 = z^2$$

$$\left(\frac{p}{x}\right)^2 + (yq)^2 = z^2$$

$$\left(\frac{1}{x} \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{1} \frac{\partial z}{\partial y}\right)^2 = z^2 \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{x \partial x}\right)^2 + \left(\frac{\partial z}{y}\right)^2 = z^2 \text{ ----- (1)} \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

Let $\partial z = \partial Z$; $x \partial x = \partial X$; $\frac{\partial y}{y} = \partial Y$

Integrating $\boxed{z = Z ; \frac{x^2}{2} = X ; \log y = Y}$

$$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = Z^2$$

$$\boxed{P^2 + Q^2 = Z^2} \text{ ----- (2)} \quad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form $F(Z, P, Q) = 0$

To find Complete Integral:

Let the complete solution of (2) is $Z = F(X + aY)$

Let $X + aY = U \Rightarrow Z = F(U)$

Then $P = \frac{dZ}{dU}$ & $Q = a \frac{dZ}{dU}$

Substitute the value of P & Q in (2)

$$(2) \Rightarrow \left(\frac{dZ}{dU}\right)^2 + \left(a \frac{dZ}{dU}\right)^2 = Z^2$$

$$\left(\frac{dZ}{dU}\right)^2 = \frac{Z^2}{(1+a^2)}$$

Taking square root on both sides

$$\frac{dZ}{dU} = \frac{Z}{\sqrt{1+a^2}}$$

$$\frac{dZ}{Z} = \frac{dU}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\log Z = \frac{1}{\sqrt{1+a^2}} U$$

$$\log Z = \frac{1}{\sqrt{1+a^2}} (X + aY)$$

This is complete integral of (2)

$$\boxed{\log z = \frac{1}{\sqrt{1+a^2}} \left(\frac{x^2}{2} + a \log y\right) + c}$$

This is the required complete integral .

Other solutions can be obtained as usual.

3. Find the complete integral of $z^2(x^2p^2 + q^2) = 1$

Solution:

Given $z^2(x^2p^2 + q^2) = 1$

This is of the form equations reducible to standard form

$$x^2p^2 + q^2 = \frac{1}{z^2}$$

$$(xp)^2 + q^2 = \frac{1}{z^2}$$

$$\left(x \frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{z^2} \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{z^2} \text{-----(1)}$$

Let $\partial z = \partial Z$; $\frac{\partial x}{x} = \partial X$; $\partial y = \partial Y$

Integrating $z = Z$; $\log x = X$; $y = Y$

$$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = \frac{1}{Z^2}$$

$$P^2 + Q^2 = \frac{1}{Z^2} \text{-----(2)} \quad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form $F(Z, P, Q) = 0$

To find Complete Integral:

Let the complete solution of (2) is $Z = F(X + aY)$

Let $X + aY = U \Rightarrow Z = F(U)$

This is of the form $F(Z, P, Q) = 0$

Then $P = \frac{dZ}{dU}$ & $Q = a \frac{dZ}{dU}$

Substitute the value of P & Q in (2)

$$(2) \Rightarrow \left(\frac{dZ}{dU}\right)^2 + \left(a \frac{dZ}{dU}\right)^2 = \frac{1}{Z^2}$$

$$\left(\frac{dZ}{dU}\right)^2 = \frac{1}{Z^2(1+a^2)}$$

Taking square root on both sides

$$\frac{dZ}{dU} = \frac{1}{Z\sqrt{1+a^2}}$$

$$ZdZ = \frac{dU}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\frac{Z^2}{2} = \frac{1}{\sqrt{1+a^2}}U$$

$$\frac{Z^2}{2} = \frac{1}{\sqrt{1+a^2}}(X + aY)$$

This is the complete integral of (2)

$$\frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}} (\log x + ay) + c$$

This is the required complete integral .
Other solutions can be obtained as usual.

4. Solve $x^2 p^2 + y^2 q^2 = z^2$

5. Solve $x^4 p^2 + y^2 zq = 2z^2$

Lagrange's Linear Differential Equations:

Equations of the form $Pp + Qq = R$ (or) $P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R$

Where P, Q, R are functions of x, y, z or constants.

Procedure :

1. Write the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2. Solve the auxiliary equation by using

- a) Method of grouping
- b) Method of multipliers

a) Method of grouping: In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$

∴ The general solution is $\phi(u, v) = 0$

b) Method of Multipliers:

i) Choose any three multipliers l, m, n which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$ then $l dx + m dy + n dz = 0$

Integrating this we get $u(x, y, z) = c_1$

ii) Choose another any three multipliers l', m', n' which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l' dx + m' dy + n' dz}{l'P + m'Q + n'R}$$

If it is possible to choose l', m', n' such that $l'P + m'Q + n'R = 0$ then $l' dx + m' dy + n' dz = 0$

Integrating this we get $v(x, y, z) = c_2$

∴ The general solution is $\phi(u, v) = 0$

1. Solve $x(y-z)p + y(z-x)q = z(x-y)$

Solution:

Given $x(y-z)p + y(z-x)q = z(x-y)$

This is of the form $Pp + Qq = R$

Where $P = x(y-z)$; $Q = y(z-x)$; $R = z(x-y)$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \text{ -----(1)}$$

i) Choose the multipliers as (1,1,1)

$$(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$= \frac{dx + dy + dz}{xy - xz + yz - xy + xz - yz} = \frac{dx + dy + dz}{0}$$

$$\therefore dx + dy + dz = 0$$

$$\text{Integrating } \int dx + \int dy + \int dz = 0$$

$$\boxed{x + y + z = c_1}$$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y-z+z-x+x-y} = \frac{dx + dy + dz}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\text{Integrating } \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \quad \Rightarrow \quad \boxed{xyz = c_2}$$

$$\therefore \text{The general solution is } \boxed{\phi(x + y + z, xyz) = 0}$$

2. **Solve** $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

Solution:

$$\text{Given } x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$$

This is of the form $Pp + Qq = R$

$$\text{Where } P = x(z^2 - y^2); Q = y(x^2 - z^2); R = z(y^2 - x^2)$$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \text{ -----(1)}$$

i) Choose the multipliers as (x, y, z)

$$(1) \Rightarrow \frac{xdx}{x^2(z^2 - y^2)} = \frac{ydy}{y^2(x^2 - z^2)} = \frac{zdz}{z^2(y^2 - x^2)}$$

$$= \frac{xdx + ydy + zdz}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2 + y^2z^2 - x^2z^2} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$$\text{Integrating } \int xdx + \int ydy + \int zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$\boxed{x^2 + y^2 + z^2 = c_1^2}$$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{\frac{dx}{x}}{\frac{x}{x}(z^2 - y^2)} = \frac{\frac{dy}{y}}{\frac{y}{y}(x^2 - z^2)} = \frac{\frac{dz}{z}}{\frac{z}{z}(y^2 - x^2)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\text{Integrating } \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \quad \Rightarrow \quad \boxed{xyz = c_2}$$

$$\therefore \text{The general solution is } \boxed{\phi(x^2 + y^2 + z^2, xyz) = 0}$$

3. Solve $(x^2 - y^2 - z^2)p + 2xyq = 2zx$

Solution:

$$\text{Given } (x^2 - y^2 - z^2)p + 2xyq = 2zx$$

This is of the form $Pp + Qq = R$

$$\text{Where } P = (x^2 - y^2 - z^2); Q = 2xy; R = 2zx$$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \quad \text{-----(1)}$$

i) by method of grouping, from last two ratios

$$(1) \Rightarrow \frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\text{Integrating } \int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1$$

$$\boxed{\frac{y}{z} = c_1}$$

ii) Choose the multipliers as (x, y, z)

$$(1) \Rightarrow \frac{xdx}{x(x^2 - y^2 - z^2)} = \frac{ydy}{y(2xy)} = \frac{zdz}{z(2zx)}$$

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

From 3rd and last ratio

$$\frac{dz}{(2zx)} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$$

Integrating $\int \frac{dz}{z} = \int \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$

$$\log z = \log(x^2 + y^2 + z^2) \quad \because \int \frac{f'(x)}{f(x)} dx = \log[f(x)]$$

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log z - \log(x^2 + y^2 + z^2) = \log c_2$$

$$\log \frac{z}{x^2 + y^2 + z^2} = \log c_2$$

$$\frac{z}{x^2 + y^2 + z^2} = c_2$$

The general solution is $\phi\left(\frac{y}{z}, \frac{z}{x^2 + y^2 + z^2}\right) = 0$

4. Solve $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

Hint:

The multipliers are (x, y, z) & $(2, 3, 4)$

The general solution is $\phi(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$

5. Solve $(y - xz)p + (yz - x)q = (x + y)(x - y)$

Hint:

The multipliers are (x, y, z) & $(y, x, 1)$

The general solution is $\phi(x^2 + y^2 + z^2, xy + z) = 0$

6. Solve $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$

Hint:

The multipliers are $\left(\frac{1}{x}, \frac{-1}{y}, \frac{1}{z}\right)$ & $(x, -y, -1)$

The general solution is $\phi\left(\frac{xz}{y}, x^2 - y^2 - 2z\right) = 0$

Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x, y) \text{ ----- (1)}$$

Let the differential operator $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 DD' + a_2 D'^2)z = f(x, y) \text{ ----- (2)}$$

The general solution of equation (2) is

$$z = \text{complementary function} + \text{Particular Integral} = \text{C.F.} + \text{P.I.}$$

To find complementary Function:

1. Write the Auxiliary equation by putting $D = m, D' = 1, z = 1, \& RHS = 0$

$$(2) \Rightarrow m^2 + a_1m + a_2 = 0$$

2. Solve the auxiliary equation, we get the roots of m . Say the roots are m_1, m_2

3. Comparing the roots of m and write the complementary function.

Case 1: The Roots are real and distinct : say $m_1 \neq m_2$

$$C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

Case 2: The Roots are real and equal : say $m_1 = m_2 = m$

$$C.F = f_1(y + mx) + xf_2(y + mx)$$

Note: If the roots are $m = \alpha \pm i\beta$

$$\text{then } C.F = f_1[y + (\alpha + i\beta)] + f_2[y + (\alpha - i\beta)]$$

To find Particular Integral :**Type : I**

If $RHS = e^{ax+by}$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

$$\text{Rule: } D = a \& D' = b$$

$$P.I = \frac{1}{f(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0$$

If Denominator = 0, then 1) multiply the numerator by x 2) differentiating denominator partially w.r.to D

$$P.I = \frac{x}{f'(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0 \because \text{ Replace } D = a \& D' = b$$

Continuing this process until we get $Dr \neq 0$.

Type : II

If $RHS = \sin(ax + by)$ (or) $RHS = \cos(ax + by)$ then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)$$

$$\text{Rule: } D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)$$

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by), \text{ Provided } Dr \neq 0$$

Note 1: After substitutions the denominator will be in terms of D & D' . Multiply and divide by D so that the denominator will have D^2 & DD' terms.

Note 2: After substitutions the denominator will be in terms of D & constant terms,

For eg. $P.I = \frac{1}{D-5} \sin(x-2y)$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D-5} \times \frac{D+5}{D+5} \sin(x-2y) = \frac{1}{D^2-25} \sin(x-2y)$$

Then apply the rule as usual.

Type : III

If $RHS = x^m y^n$ (polynomial type) then

$P.I = \frac{1}{f(D, D')} x^m y^n$, we bring this into a standard binomial format, by taking out highest power term of D.

$$(i.e) P.I = [1 \pm f(D, D')]^{-1} x^m y^n$$

This will be expanded by using the formulae

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Note:

$$\frac{1}{D} = \int dx, D' = \frac{\partial}{\partial y}$$

Type : IV (Exponential shifted rule)

If $RHS = e^{ax+by} \cos(ax+by)$ (or) $RHS = e^{ax+by} \sin(ax+by)$ (or) $RHS = e^{ax+by} x^m y^n$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by} \sin(ax+by)$$

$$P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)} \sin(ax+by)$$

Here after apply the rule as we discussed in Type II&III

Type : V

If $RHS = y \cos x$ (or) $RHS = y \sin x$ then

Case 1:

$$P.I = \frac{1}{D - mD'} y \cos x$$

$$P.I = \frac{1}{D - mD'} \int (c - mx) \cos x dx \quad \because \text{Rule: } y = c - mx$$

Case 2:

$$P.I = \frac{1}{D + mD'} y \cos x$$

$$P.I = \frac{1}{D + mD'} \int (c + mx) \cos x dx \quad \because \text{Rule: } y = c - mx$$

Note: After integration we have to replace $c - mx = y$

1. **Solve** $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

Solution:

Given $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

To find C.F

The auxiliary equation is

$$m^2 - m - 20 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 5)(m + 4) = 0$$

$$m = 5, -4$$

$$\therefore C.F = f_1(y - 4x) + f_2(y + 5x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + mx) + x f_2(y + mx)$$

To find P.I

$$P.I = \frac{1}{D^2 - DD' - 20D'^2} [e^{5x+y} + \sin(4x - y)]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1 \quad \text{Type:1}$$

$$= \frac{1}{25 - 5 - 20} e^{5x+y} = \frac{1}{0} e^{5x+y} \quad \therefore \text{Introducing } x \text{ in Nr.Diff Dr, partially w.r.to } D$$

$$= \frac{x}{2D - D'} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1$$

$$= \frac{x}{10 - 1} e^{5x+y}$$

$$P.I_1 = \frac{x}{9} e^{5x+y}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y) \quad \text{here } a = 4 \text{ \& } b = -1 \quad (\text{Type:2})$$

Rule: replace $D^2 = -(a^2) = -16$; $D'^2 = -(b^2) = -1$ & $DD' = -(ab) = 4$

$$P.I_2 = \frac{1}{-16 - 4 - 20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$$

$$= \frac{x}{2D - D'} \sin(4x - y) \quad \therefore \text{Introducing } x \text{ in Nr.Diff Dr, partially w.r.to } D$$

$$= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x - y)$$

$$= \frac{x D}{2D^2 - DD'} \sin(4x - y)$$

$$= \frac{x D}{2(-16) - 4} \sin(4x - y)$$

$$= \frac{x D (\sin(4x - y))}{-32 - 4}$$

$$P.I_2 = \frac{4x \cos(4x + 3)}{-36} = \frac{-1}{9} x \cos(4x + 3) \quad \therefore \frac{d}{dx} (\sin nx) = n \cos nx$$

$$P.I_2 = \frac{-1}{9} x \cos(4x + 3)$$

$$P.I = \frac{x}{9} e^{5x+y} - \frac{1}{9} x \cos(4x + 3) = \frac{x}{9} [e^{5x+y} - \cos(4x + 3)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - 4x) + f_2(y + 5x) + \frac{x}{9} [e^{5x+y} - \cos(4x + 3)]$$

2. Solve $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$

Solution: same as previous problem

Hint:

$$\text{Given } \frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$$

$$(D^3 - 2D^2 D') z = e^{x+2y} + 4 \sin(x + y) \quad \therefore D = \frac{\partial}{\partial x} \quad \& \quad D' = \frac{\partial}{\partial y}$$

$$m = 0, 0, 2$$

$$C.F = f_1(y) + x f_2(y) + f_2(y + 2x)$$

$$P.I = \frac{-1}{3} e^{x+2y} - 4 \cos(x+2y)$$

3. Solve $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

Solution:

Given $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

To find C.F

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2, -3$$

$$\therefore C.F = f_1(y-3x) + f_2(y+2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} [x^2y + e^{3x+y}]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad (\text{Type:3})$$

$$= \frac{1}{D^2 \left[1 + \left(\frac{DD' - 6D'^2}{D^2} \right) \right]} x^2y$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{D' - 6D'^2}{D} - \frac{6D'^2}{D^2} \right) \right]^{-1} x^2y$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{D' - 6D'^2}{D} - \frac{6D'^2}{D^2} \right) + \dots \right] x^2y \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{D^2} \left[1 - \frac{D'}{D} + \frac{6D'^2}{D^2} \right] x^2y \quad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2} \left[x^2y - \frac{D'(x^2y)}{D} + \frac{6D'^2(x^2y)}{D^2} \right]$$

$$= \frac{1}{D^2} \left[x^2y - \frac{x^2}{D} \right]$$

$$= \frac{1}{D^2} \left[x^2y - \int x^2 dx \right] \quad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2} \left[x^2y - \frac{x^3}{3} \right]$$

$$= \frac{1}{D} \int \left(x^2y + \frac{x^3}{3} \right) dx$$

$$= \int \left(\frac{x^3y}{3} - \frac{x^4}{12} \right) dx$$

$$P.I_1 = \frac{x^4y}{12} - \frac{x^5}{60}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y} \quad \text{Rule: replace } D = a = 3 \text{ \& } D' = b = 1 \quad \text{Type:1}$$

$$= \frac{1}{9+3-6(1)} e^{3x+y} = \frac{1}{6} e^{3x+y}$$

$$P.I_2 = \frac{1}{6} e^{3x+y}$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

4. Solve $(D^2 + 2DD' + D'^2)z = x^2 y + e^{x-y}$

Solution: same as previous problem

Hint:

$$m = -1, -1$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

5. Solve $(D^2 - 6DD' + 5D'^2)z = xy + e^x \sinh y$

Solution:

$$\text{Given } (D^2 - 6DD' + 5D'^2)z = xy + e^x \sinh y$$

$$= xy + e^x \left(\frac{e^y - e^{-y}}{2} \right) \quad \because \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}, \text{ here } \theta = y$$

$$= xy + \left(\frac{e^x e^y - e^x e^{-y}}{2} \right)$$

$$(D^2 - 6DD' + 5D'^2)z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \quad \because e^a e^b = e^{a+b}$$

To find C.F

The auxiliary equation is

$$m^2 - 6m + 5 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-1)(m-5) = 0$$

$$m = 1, 2$$

$$\therefore C.F = f_1(y+x) + f_2(y+5x) \quad \because \text{The roots are real and distinct } \Rightarrow C.F = f_1(y+m_1 x) + x f_2(y+m_2 x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2} \left[xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \right]$$

$$P.I = P.I_1 + P.I_2 - P.I_3 \quad \text{----- (1)}$$

To find P.I₁

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} xy \quad (\text{Type:3})$$

$$= \frac{1}{D^2 \left[1 + \left(\frac{-6DD' + 5D'^2}{D^2} \right) \right]} xy$$

$$\begin{aligned}
&= \frac{1}{D^2} \left[1 + \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2} \right) \right]^{-1} xy \\
&= \frac{1}{D^2} \left[1 - \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2} \right) + \dots \right] xy \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \\
&= \frac{1}{D^2} \left[1 + \frac{6D'}{D} - \frac{5D'^2}{D^2} + \dots \right] xy \\
&= \frac{1}{D^2} \left[xy + \frac{6x}{D} - \frac{5(0)}{D^2} + \dots \right] xy \quad \because D'(y) = \frac{\partial}{\partial y}(y) = 1 \text{ \& } D'^2(y) = \frac{\partial^2}{\partial y^2}(y) = 0 \\
&= \frac{1}{D^2} \left[xy + 6 \int x dx \right] = \frac{1}{D^2} \left[xy + \frac{6x^2}{2} \right] \quad \because \frac{1}{D} = \int dx \\
&= \frac{1}{D} \int (xy + 3x^2) dx = \frac{1}{D} \left(\frac{x^2 y}{2} + \frac{3x^3}{3} \right) = \int \left(\frac{x^2 y}{2} + x^3 \right) dx
\end{aligned}$$

$$P.I_1 = \frac{x^3 y}{6} + \frac{x^4}{4}$$

To find P.I₂

$$\begin{aligned}
P.I_2 &= \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x+y}}{2} \quad \text{Type:1} \\
&= \left(\frac{1}{2} \right) \frac{1}{1-6+5} e^{x+y} = \left(\frac{1}{2} \right) \frac{1}{0} e^{x+y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = 1 \\
&= \left(\frac{1}{2} \right) \frac{x}{2D - 6D' + 0} e^{x+y} \\
&= \left(\frac{1}{2} \right) \frac{x}{2-6} e^{x+y} \\
&= \left(\frac{1}{2} \right) \frac{-x}{4} e^{x+y}
\end{aligned}$$

$$P.I_2 = \frac{-x}{8} e^{x+y}$$

To find P.I₃

$$\begin{aligned}
P.I_3 &= \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x-y}}{2} \quad \text{Type:1} \\
&= \left(\frac{1}{2} \right) \frac{1}{1+6+5} e^{x+y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = -1
\end{aligned}$$

$$P.I_3 = \frac{1}{24} e^{x+y}$$

$$(1) \Rightarrow P.I = P.I_1 + P.I_2 - P.I_3 = \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + f_2(y+5x) + \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

6. Solve $(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$

Solution: same as previous problem

Hint:

$$(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$$

$$(D^2 + 2DD' + D'^2)z = \frac{e^{x+y} - e^{-(x-y)}}{2} + e^{x+2y} = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$$

$$(D^2 + 2DD' + D'^2)z = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$$

$$m = -1, -1$$

$$C.F = f_1(y-x) + xf_2(y-x)$$

$$P.I = \frac{e^{x+y}}{8} - \frac{e^{-x-y}}{8} - \frac{e^{x+2y}}{9}$$

7.

$$\text{Solve } (D^2 - 4DD' + 4D'^2)z = e^{x+2y}$$

Solution:

$$\text{Hint: } m = 2, 2$$

$$C.F = f_1(y+x) + f_2(y+5x)$$

$$P.I = \frac{x^2}{2} e^{2x+y}$$

8.

$$\text{Solve } (2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y)$$

Solution:

$$\text{Given } (2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y)$$

To find C.F

The auxiliary equation is

$$2m^2 - 5m + 2 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{here } a = 2, b = -5, c = 2$$

$$= \frac{-(-5) \pm \sqrt{25 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$$m = \frac{5+3}{4}, m = \frac{5-3}{4} \Rightarrow \boxed{m = 2, m = \frac{1}{2}}$$

$$\therefore \boxed{C.F = f_1(y+2x) + f_2\left(y + \frac{x}{2}\right)} \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x+y) \quad \text{Type:2}$$

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x+y) \quad \text{here } a = 2 \& b = 1$$

$$\text{Rule: replace } D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \& DD' = -(ab) = -2$$

$$P.I = \frac{1}{2(-4) - 5(-2) + 2(-1)} 5 \sin(2x+y)$$

$$P.I = \frac{1}{-8+10-2} 5 \sin(2x+y) = \frac{1}{0} 5 \sin(2x+y)$$

$$\begin{aligned}
&= \frac{x}{4D - 5D' + 0} 5 \sin(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff Dr. partially w.r.to } D\} \\
&= \frac{x}{4D - 5D'} \times \frac{D}{D} 5 \sin(2x + y) \\
&= 5 \frac{x D}{4D^2 - 5DD'} \sin(2x + y) \\
&= 5 \frac{x D}{4(-4) - 5(-2)} \sin(2x + y) = 5 \frac{x D [\sin(2x + y)]}{-16 + 10} \\
&= \frac{5}{-6} x [2 \cos(2x + y)]
\end{aligned}$$

$$P.I = \frac{-5}{3} x \cos(2x + y)$$

The general solution is
 $z = C.F + P.I$

$$z = f_1(y + 2x) + f_2\left(y + \frac{x}{2}\right) - \frac{5}{3} x \cos(2x + y)$$

9. Solve the equation $(D^3 + D^2 D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$

Solution:

$$\text{Given } (D^3 + D^2 D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$$

To find C.F

The auxiliary equation is

$$m^3 + m^2 - 4m - 4 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$m^2(m + 1) - 4(m + 1) = 0$$

$$(m + 1)(m^2 - 4) = 0$$

$$m = -1, -2, 2$$

$$\therefore C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x)$$

To find P.I

$$P.I = \frac{1}{D^3 + D^2 D' - 4DD'^2 - 4D'^3} \cos(2x + y) \quad \text{here } a = 2, b = 1 \quad (\text{Type:2})$$

$$\text{Rule: replace } D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \& DD' = -(ab) = -2$$

$$P.I = \frac{1}{-4D - 4D' - 4D(-1) - 4(-1)D'} \cos(2x + y)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x + y) = \frac{1}{0} \cos(2x + y)$$

$$P.I = \frac{x}{3D^2 + 2DD' - 4D'^2 - 0} \cos(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff Dr. partially w.r.to } D\}$$

$$= \frac{x}{3(-4) + 2(-2) - 4(-1)} \cos(2x + y) = \frac{x}{-12 - 4 + 4} \cos(2x + y)$$

$$P.I = \frac{-x}{12} \cos(2x + y)$$

The general solution is
 $z = C.F + P.I$

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x) - \frac{x}{12} \cos(2x + y)$$

10. Solve $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

Solution:

Given $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

To find C.F

The auxiliary equation is

$m^3 - 7m - 6 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0

$m = -1, m^2 - m - 6 = 0$

$m = -1, (m - 3)(m + 2) = 0$

$m = -1, -2, 3$

$\therefore C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

To find P.I

$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3} [\cos(x + 2y) + 4]$

$P.I = P.I_1 + P.I_2$

$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cos(x + 2y) \quad \text{here } a = 1, b = 2 \text{ (Type:2)}$

Rule: replace $D^2 = -(a^2) = -1; D'^2 = -(b^2) = -4$ & $DD' = -(ab) = -2$

$$\begin{aligned}
 P.I_1 &= \frac{1}{(-1)D - 7D(-2) - 6(-2)D'} \cos(x + 2y) = \frac{1}{-D + 14D + 12D'} \cos(x + 2y) \\
 &= \frac{1}{13D + 12D'} \times \frac{D}{D} \cos(x + 2y) \\
 &= \frac{D}{13D^2 + 12D'D} \cos(x + 2y) \\
 &= \frac{D}{13(-1) + 12(-2)} \cos(x + 2y) = \frac{D \cos(x + 2y)}{-13 - 24}
 \end{aligned}$$

$P.I_1 = \frac{-\sin(x + 2y)}{37}$

To find P.I₂

$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} \quad \therefore e^0 = 1 \quad \text{Type:1}$

$= \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} = \frac{1}{0} 4e^{0x+0y} \quad \text{Rule: Replace } D = 0, D' = 0$

$= \frac{x}{3D^2 - 7D'^2 - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$

$= \frac{x}{6D - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$

$= \frac{x^2}{6} 4$

$= \frac{x^2}{6}$

$$P.I_2 = \frac{2x^2}{3}$$

$$P.I = \frac{-1}{37} \sin(x+2y) + \frac{2x^2}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37} \sin(x+2y) + \frac{2x^2}{3}$$

11. Solve $(D^2 + 3DD' - 4D'^2)z = xy + \cos(2x+y)$

Solution: same as previous problem

Hint:

$$m = -4, 1$$

$$C.F = f_1(y-4x) + xf_2(y+x)$$

$$P.I = \frac{-1}{6} \cos(2x+y) + \frac{x^3y}{6} - \frac{x^4}{8}$$

12. Solve $(D^2 + 3DD' - 4D'^2)z = x + \sin y$

Hint:

$$m = -4, 1$$

$$C.F = f_1(y-4x) + xf_2(y+x)$$

$$P.I = \frac{1}{D^2 + 3DD' - 4D'^2} [x + \sin(0x+y)] = \dots = \frac{x^3}{6} + \frac{\sin y}{4}$$

13. Solve $(D^2 - DD' - 2D'^2)z = (2x+3y) + e^{3x+4y}$

Hint:

$$m = -1, 2$$

$$C.F = f_1(y-x) + f_2(y+2x)$$

$$P.I = \frac{5x^3}{6} + \frac{3x^2y}{2} + \frac{1}{35} e^{3x+4y}$$

14. Solve $(D^2 - 2DD' + D'^2)z = (2+4x)e^{x+2y}$

Solution:

$$\text{Given } (D^2 - 2DD' + D'^2)z = x^2y^2e^{x+2y}$$

To find C.F

The auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$\therefore C.F = f_1(y+x) + xf_2(y+x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} (2+4x)e^{x+2y} \quad \text{here } a = 2, b = 1 \quad (\text{Type:4})$$

$$P.I = \frac{1}{(D-D')^2} (2+4x)e^{x+2y}$$

Rule: replace $D = D + a = D + 1$; $D' = D' + b = D' + 2$

$$P.I = e^{x+2y} \frac{1}{(D+1-D'-2)^2} (2+4x)$$

$$\begin{aligned}
&= e^{x+2y} \frac{1}{(D-D'-1)^2} (2+4x) \\
&= e^{x+2y} \frac{1}{[-(1-D+D')]^2} (2+4x) = e^{x+2y} \frac{1}{[1-(D-D')]^2} (2+4x) \\
&= e^{x+2y} [1-(D-D')]^{-2} (2+4x) \\
&= e^{x+2y} [1+2(D-D')+3(D-D')^2+\dots] (2+4x) \quad \because (1-x)^{-2} = 1+2x+3x^2+4x^3+\dots \\
&= e^{x+2y} [1+2D-2D'+3(D^2-2DD'+D'^2)+\dots] (2+4x) \\
&= e^{x+2y} [1+2D-2D'+3D^2-6DD'+3D'^2] (2+4x) \\
&= e^{x+2y} [1+2D+3D^2] (2+4x) \quad \because \text{there is no } y \text{ term in RHS, neglect the term } D' \\
&= e^{x+2y} [(2+4x)+2D(2+4x)+3D^2(2+4x)] \\
&= e^{x+2y} [2+4x+2(4)+0]
\end{aligned}$$

$$P.I = e^{x+2y} [4x+10]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y} [4x+10]$$

15. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x+3y)$

Solution:

Given $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x+3y)$

$$(D^2 - D'^2)z = e^{x-y} \sin(2x+3y)$$

To find C.F

The auxiliary equation is

$$m^2 - 1 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$m^2 = 1 \Rightarrow m = -1, 1$$

$$\therefore C.F = f_1(y-x) + f_2(y+x)$$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2} e^{x-y} \sin(2x+3y) \quad \text{here } a=1, b=-1 \quad (\text{Type:4})$$

Rule: replace $D = D + a = D + 1; D' = D' + b = D' - 1$

$$P.I = e^{x-y} \frac{1}{(D+1)^2 - (D'-1)^2} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{D^2 + 2D - D'^2 + 2D'} \sin(2x+3y) \quad \text{Here } a=2, b=3$$

Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -9$ & $DD' = -(ab) = -6$

$$= e^{x-y} \frac{1}{-4 + 2D - (-9) + 2D'} \sin(2x+3y) = e^{x-y} \frac{1}{2D + 2D' + 5} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{2D + 2D' + 5} \times \frac{D}{D} \sin(2x+3y)$$

$$= e^{x-y} \frac{D}{2D^2 + 2DD' + 5D} \sin(2x + 3y)$$

$$= e^{x-y} \frac{D}{-8 - 12 + 5D} \sin(2x + 3y) = e^{x-y} \frac{D}{5D - 20} \sin(2x + 3y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= e^{x-y} \frac{D}{5D - 20} \times \frac{5D + 20}{5D + 20} \sin(2x + 3y)$$

$$= e^{x-y} \frac{5D^2 + 20D}{25D^2 - 400} \sin(2x + 3y)$$

$$= e^{x-y} \frac{5D^2 \sin(2x + 3y) + 20D \sin(2x + 3y)}{25(-4) - 400}$$

$$= e^{x-y} \frac{5D \cos(2x + 3y) \times 2 + 20 \cos(2x + 3y) \times 2}{-100 - 400}$$

$$= e^{x-y} \frac{-20 \sin(2x + 3y) + 40 \cos(2x + 3y)}{-500}$$

$$P.I = \frac{e^{x-y}}{25} [\sin(2x + 3y) - 2 \cos(2x + 3y)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - x) + f_2(y + x) + \frac{e^{x-y}}{25} [\sin(2x + 3y) - 2 \cos(2x + 3y)]$$

16. Solve $(D^2 + DD' - 6D'^2)z = y \cos x$

Solution:

$$\text{Given } (D^2 + DD' - 6D'^2)z = y \cos x$$

To find C.F

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m + 3) = 0$$

$$m = 2, -3$$

$$\therefore C.F = f_1(y - 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct } \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x \quad \text{Type: 5}$$

$$= \frac{1}{(D + 3D')(D - 2D')} y \cos x$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x \, dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D + 3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)] \quad \because \int uv \, dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D + 3D')} [y \sin x - 2 \cos x]$$

$$= \frac{1}{(D + 3D')} \int [(c + 3x) \sin x - 2 \cos x] \, dx \quad \because \text{Rule: } y = c + mx \text{ here } m = 3$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2 \sin x \quad \because y = c + 3x$$

$$= -y \cos x + 3 \sin x - 2 \sin x$$

$$P.I = \sin x - y \cos x$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$$

17. Solve $(D^2 - 5DD' + 6D'^2)z = y \sin x$

Solution:

Given $(D^2 - 5DD' + 6D'^2)z = y \sin x$

To find C.F

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m - 3) = 0$$

$$m = 2, 3$$

$$\therefore C.F = f_1(y + 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} y \sin x \quad \text{Type: 5}$$

$$= \frac{1}{(D - 3D')(D - 2D')} y \sin x$$

$$= \frac{1}{(D - 3D')} \int (c - 2x) \sin x dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D - 3D')} [(c - 2x)(-\cos x) - (-2)(-\sin x)] \quad \because \int uv dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D - 3D')} [-y \cos x - 2 \sin x] = \frac{-1}{(D - 3D')} [y \cos x + 2 \sin x]$$

$$= -\int [(c - 3x) \cos x + 2 \sin x] dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 3$$

$$= -[(c - 3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)] \quad \because y = c - 3x$$

$$= -[y \sin x - 3 \cos x - 2 \cos x]$$

$$P.I = 5 \cos x - y \sin x$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + 3x) + f_2(y + 2x) + 5 \cos x - y \sin x$$

Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} + a_3 \frac{\partial z}{\partial x} + a_4 \frac{\partial z}{\partial y} = f(x, y) \quad \text{----- (1)}$$

Let the differential operator $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 DD' + a_2 D'^2 + a_3 D + a_4 D')z = f(x, y) \quad \text{----- (2)}$$

The general solution of equation (2) is

$$z = \text{complementary function} + \text{Particular Integral} = C.F + P.I$$

To find complementary Function:

Case : I

The given PDE will bring into the form of $(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0$

$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

Case : II

The given PDE will bring into the form of $(D - mD' - C)^2 z = 0$

$$C.F = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx)$$

Note: Particular Integral can be obtained, similar like in Homogeneous types.

1. Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

Solution:

Given $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

To find C.F

$$((D + D')^2 - 2(D + D'))z = 0$$

$$(D + D')(D + D' - 2)z = 0 \quad \text{----- (1)}$$

This is of the form

$$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 \quad \text{----- (2)}$$

Comparing (1) & (2)

$$m_1 = -1, C_1 = 0, m_2 = -1, C_2 = 2.$$

$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

$$C.F = f_1(y - x) + e^{2x} f_2(y - x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} (e^{3x+y} + \sin(x + 2y))$$

$$P.I = P.I_1 + P.I_2$$

To find P.I₁

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{3x+y} \quad \text{Rule: replace } D = 3 \text{ \& } D' = 1 \text{ Type:1}$$

$$= \frac{1}{9 + 6 + 1 - 6 - 2} e^{3x+y}$$

$$P.I_1 = \frac{1}{8} e^{3x+y}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \quad \text{here } a = 1, b = 2 \text{ type:2}$$

Rule: replace $D^2 = -(a^2) = -1$; $D'^2 = -(b^2) = -4$ & $DD' = -(ab) = -2$

$$P.I_2 = \frac{1}{-1 + 2(-2) - 4 - 2D - 2D'} \sin(x + 2y) = \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{-1}{2D + 2D' + 9} \times \frac{D}{D} \sin(x + 2y)$$

$$= \frac{-D}{2D^2 + 2D'^2 + 9D} \sin(x + 2y)$$

$$= \frac{-D}{-2 - 8 + 9D} \sin(x + 2y) = \frac{-D}{9D - 10} \sin(x + 2y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= \frac{-D}{9D - 10} \times \frac{9D + 10}{9D + 10} \sin(x + 2y)$$

$$= \frac{-9D^2 - 10}{81D^2 - 100} \sin(x+2y)$$

$$= \frac{-9D^2 \sin(x+2y) - 10D \sin(x+2y)}{-81 - 100}$$

$$= \frac{1}{-181} [-9D \cos(x+2y) - 10 \cos(x+2y)]$$

$$P.I_2 = \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

$$P.I = \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + e^{2x} f_2(y-x) + \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

2. Solve $(D^2 - D'^2 - 3D + 3D')z = e^{3x+y} + 4$

Solution:

Given $(D^2 - D'^2 - 3D + 3D')z = e^{3x+y} + 4$

To find C.F

$$((D + D')(D - D') - 3(D - D'))z = 0$$

$$(D - D')(D + D' - 3)z = 0 \quad \text{----- (1)}$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 \quad \text{----- (2)}$$

Comparing (1) & (2)

$$m_1 = 1, C_1 = 0, m_2 = -1, C_2 = 3.$$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

$$C.F = f_1(y+x) + e^{3x} f_2(y-x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'} (e^{3x+y} + 4)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - D'^2 - 3D + 3D'} e^{3x+y} \quad \text{here } a = 3, b = 1 \quad \text{type : 1}$$

$$= \frac{1}{9 - 1 - 9 + 3} e^{3x+y} \quad \text{Rule: Replace } D = 3, D' = 1$$

$$P.I_1 = \frac{1}{2} e^{3x+y}$$

$$P.I_2 = \frac{1}{D^2 - D'^2 - 3D + 3D'} 4e^{0x+0y} \quad \text{here } a = 0, b = 0 \quad \text{type : 1}$$

$$= \frac{1}{0} 4e^{0x+0y} \quad \text{Rule: Replace } D = 0, D' = 0$$

Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step

$$= \frac{x}{2D - 0 - 3 + 0} 4e^{0x+0y}$$

$$= \frac{4x}{-3}$$

$$P.I_2 = \frac{-4x}{3}$$

$$P.I = \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

3. Solve $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$

Solution:

$$\text{Given } (2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$$

To find C.F

$$(2D^2 - DD' - D'^2 + 6D + 3D')z = 0$$

$$(2D + D')(D - D') + 3(2D + D')z = 0$$

$$(2D + D')(D - D' + 3)z = 0$$

$$\left(D + \frac{D'}{2}\right)(D - D' + 3)z = 0$$

This is of the form

$$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 \text{----- (2)}$$

Comparing (1) & (2)

$$m_1 = \frac{-1}{2}, C_1 = 0, m_2 = 1, C_2 = -3.$$

$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

$$C.F = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y+x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \quad \text{Type : 4}$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^{0x+y} \quad \text{here } a = 0, b = 1$$

Rule: replace $D = D + a = D + 0 = D$; $D' = D' + b = D' + 1$

$$P.I = e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x$$

$$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \quad \text{Type : 3}$$

$$= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x$$

$$= e^y \frac{1}{2 \left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) \right]} x$$

[normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$= \frac{e^y}{2} \left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) \right]^{-1} x \quad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

$$= \frac{e^y}{2} \left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) + \dots \right] x \quad [\text{neglect the terms } D', \text{ since } D'(x) = 0]$$

$$= \frac{e^y}{2} \left[1 - \left(\frac{5D}{2} \right) + \dots \right] x \quad \because D^2(x) = 0$$

$$= \frac{e^y}{2} \left[x - \frac{5(1)}{2} \right] \quad \because D^2(x) = 0$$

$$= \frac{e^y}{2} \left[\frac{2x - 5}{2} \right]$$

$$P.I = \frac{e^y}{4} [2x - 5]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{e^y}{4} [2x - 5]$$