ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire

function.

An entire function is analytic everywhere except at $z = \infty$.

Example: e^z , sin z, cos z, sinhz, cosh z

Example: Show that $f(z) = \log z$ analytic everywhere except at the origin and find its

derivatives.

Solution:

Let
$$z = re^{i\theta}$$

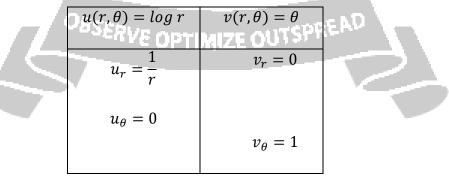
 $f(z) = \log z$
 $= \log(re^{i\theta}) = \log r + \log(e^{i\theta}) = \log r + \log(e^{i\theta})$

But, at the origin, r = 0. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, f(z) is not defined at the origin and hence is not differentiable there.

At points other than the origin, we have



So, *logz* satisfies the C–R equations.

Further $\frac{1}{r}$ is not continuous at z = 0.

So, u_r , u_θ , v_r , v_θ are continuous everywhere except at z = 0. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

Note :
$$e^{-\infty} = 0$$

 $\log e^{-\infty} = \log 0$; $-\infty = \log 0$

iθ

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

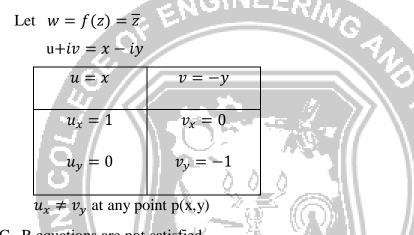
Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.) e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}$$

Example: Check whether $w = \overline{z}$ is analytics everywhere.

Solution:



Hence, C-R equations are not satisfied.

: The function f(z) is nowhere analytic.

Example: Test the analyticity of the function w = sin z. Solution:

Let w = f(z) = sinz

 $u + iv = \sin(x + iy)$, KANYA^{KUN}

 $u + iv = \sin x \cos iy + \cos x \sin iy$

$$u + iv = \sin x \cosh y + i \cos x \sin hy$$

Equating real and imaginary parts, we get OUTSPREND

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$
$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

 $\therefore u_x = v_y$ and $u_y = -v_x$

C – R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not. Solution:

Let
$$f(z) = 2xy + i(x^2 - y^2)$$

(*i.e.*)
 $u = 2xy$ $v = x^2 - y^2$
 $\frac{\partial u}{\partial x} = 2y$ $\frac{\partial v}{\partial x} = 2x$
 $\frac{\partial u}{\partial y} = 2x$ $\frac{\partial v}{\partial y} = -2y$

C-R equations are not satisfied.

Hence, f(z) is not an analytic function.

Example: Prove that $f(z) = \cosh z$ is an analytic function and find its derivative. Solution:

Given
$$f(z) = \cosh z = \cos(iz) = \cos[i(x + iy)]$$

= $\cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$
 $u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$	
$u_x = \sinh x \cos y$ $u_y = -\cosh x \sin y$	$v_x = \cosh x \sin y$ $v_y = \sinh x \cos y$	NA N

 u_x, u_y, v_x and v_y exist and are

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continuous.

$$u_x = v_y$$
 and $u_y = -v_x$ OPTIMIZE OUTSPRE

C-R equations are satisfied.

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 \therefore f(z) is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

 $= \sinh x \cos y + i \cosh x \sin y$

 $= \sinh(x + iy) = \sinh z$

Example: If w = f(z) is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where z = x + iy, and

prove that $\frac{\partial^2 w}{\partial z \partial \overline{z}} = 0.$

Solution:

Let w = u(x, y) + iv(x, y)

As f(z) is analytic, we have $u_x = v_y$, $u_y = -v_x$

Now,
$$\frac{dw}{dz} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u + iv) = -i\frac{\partial}{\partial y}(u + iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$
We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \overline{z}} = 0$$
Also
$$\frac{\partial^2 w}{\partial \overline{z} \partial z} = 0$$

Example: Prove that every analytic function w = u(x, y) + iv(x, y)can be expressed as a function of z alone.

Proof:

Let
$$z = x + iy$$
 and $\overline{z} = x - iy$
 $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z + \overline{z}}{2i}$

Hence, u and v and also w may be considered as a function of z and \overline{z}

Consider
$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + i \frac{\partial v}{\partial \overline{z}}$$

$$= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}}\right) + \left(\frac{\partial v}{\partial x} \frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \overline{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$
The product of the second se

This means that w is independent of \overline{z}

(i.e.) w is a function of z alone.

This means that if w = u(x, y) + iv(x, y) is analytic, it can be rewritten as a function of (x + iy).

Equivalently a function of \overline{z} cannot be an analytic function of z.

Example: Find the constants a, b, c if f(z) = (x + ay) + i(bx + cy) is analytic. Solution:

$$f(z) = u(x, y) + iv(x, y)$$

= (x + ay) + i(bx + cy)		
u = x + ay	v = bx + cy	
$u_x = 1$	$v_x = b$	
$u_y = a$	$v_y = c$	

Given f(z) is analytic

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x \text{NEER}$$
$$1 = c \quad \text{and} \quad a = -b$$

Example: Examine whether the following function is analytic or not $f(z) = e^{-x}(\cos y - i \sin y)$.

Solution:

Given
$$f(z) = e^{-x}(\cos y - i \sin y)$$

 $\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$
 $u = e^{-x} \cos y$ $v = -e^{-x} \sin y$
 $u_x = -e^{-x} \cos y$ $v_x = e^{-x} \sin y$
 $u_y = -e^{-x} \sin y$ $v_y = -e^{-x} \cos y$
Here, $u_x = v_y$ and $u_y = -v_x$
 \Rightarrow C-R equations are satisfied
 $\Rightarrow f(z)$ is analytic.

Example: Test whether the function $f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}(\frac{y}{x}))$ is analytic or not. Solution:

Given
$$f(z) = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$$
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(*i.e.*) $u + iv = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$
 $u = \frac{1}{2}\log(x^2 + y^2)$ $v = \tan^{-1}\left(\frac{y}{x}\right)$

$$u_{x} = \frac{1}{2} \frac{1}{x^{2} + y^{2}} (2x)$$

$$v_{x} = \frac{1}{1 + \frac{y^{2}}{x^{2}}} \left[-\frac{y}{x^{2}} \right]$$

$$= \frac{x}{x^{2} + y^{2}}$$

$$u_{y} = \frac{1}{2} \frac{1}{x^{2} + y^{2}} (2y)$$

$$u_{y} = \frac{1}{2} \frac{1}{x^{2} + y^{2}} (2y)$$

$$u_{y} = \frac{1}{1 + \frac{y^{2}}{x^{2}}} \left[\frac{1}{x} \right]$$

$$u_{y} = \frac{y}{x^{2} + y^{2}}$$

$$u_{y} = \frac{1}{1 + \frac{y^{2}}{x^{2}}} \left[\frac{1}{x} \right]$$

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Example: Find where each of the following functions ceases to be analytic.

(i)
$$\frac{z}{(z^2-1)}$$
 (ii) $\frac{z+i}{(z-i)^2}$
Solution:
(i) Let $f(z) = \frac{z}{(z^2-1)}$
 $f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$
 $f(z)$ is not analytic, where $f'(z)$ does not exist.
(*i.e.*) $f'(z) \rightarrow \infty$
(*i.e.*) $(z^2-1)^2 = 0$
(*i.e.*) $(z^2-1)^2 = 0$
(*i.e.*) $z^2-1=0$
OBSERV $z = 1$
 $z = \pm 1$ IMIZE OUTSPREAD

 $\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let
$$f(z) = \frac{z+i}{(z-i)^2}$$

 $f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$
 $f'(z) \to \infty, at \ z = i$

 $\therefore f(z)$ is not analytic at z = i.