CHARACTERISTIC EQUATION

If A is any square matrix of order n, the matrix $A - \lambda I$ where I is the unit matrix and λ be scaler of order n can be formed as

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$ is called the characteristic equation of A.

Working Rule for Characteristic Equation

Type I: For 2×2 matrix

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then the characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

Where s_1 =Sum of the leading diagonal elements $= a_{11} + a_{22}$

$s_2 = |A|$ =Determinant of a matrix A.

Type II: For 3×3 matrix

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then the characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

Where s_1 =Sum of the leading diagonal elements $= a_{11} + a_{22} + a_{33}$

s_2 =Sum of minors of leading diagonal elements

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$s_3 = |A|$ = Determinant of a matrix A.

Example: Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 = sum of the main diagonal element

$= 1 + 2 = 3$

$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$

Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-5} & \mathbf{4} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 = sum of the main diagonal element

$= 1 + 4 = 5$

$$s_2 = |A| = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6$$

Characteristic equation is $\lambda^2 - 5\lambda - 6 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 2 + 2 = 6$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 11$$

$$s_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2(4 - 0) - 0 + 1(0 - 2)$$

$= 8 - 2 = 6$

Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 2 + 1 = 5$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 7$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(2 - 0) - 1(1 - 0) + 1(0 - 0)$$

$= 4 - 1 = 3$

Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Example: Find the characteristic polynomial of the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic polynomial is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$

s_1 = sum of the main diagonal element

$= 0 + 1 + 2 = 3$

s_2 = sum of the minors of the main diagonalelement

$= \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$s_3 = |A| = \begin{vmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{vmatrix} = 0 + 2(-2+2) - 2(1+1)$

$= -4$

Characteristic polynomial is $\lambda^3 - 3\lambda^2 + 4$