2.3 CONTINUITY EQUATION

Rate of flow or Discharge (Q)

It is defined as the quantity of a fluid flowing per second through a section of pipe or channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of the liquid flowing cross the section per second. or compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section.

Thus i) For liquids the unit of Q is $m3/sec$ or Litres/sec.

ii) For gases the unit of Q is Kg f/sec or Newton/sec.

The discharge $Q = A \times V$

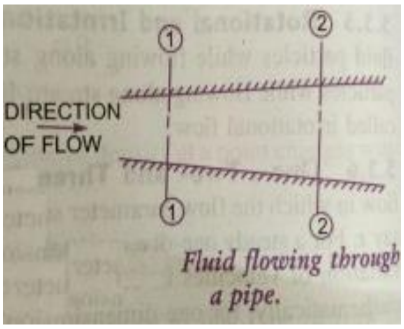
Where, A = Area of cross-section of pipe.

V= Average velocity of fluid across the section.

CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called Continuity equation. Thus for a fluid flowing through the pipe at all cross- sections, the quantity of fluid per second is constant. Consider two cross- sections of a pipe.

Let $V1$ = Average velocity at cross- section 1-1
$\rho 1$ = Density of fluid at section 1-1
$A1$ = Area of pipe at section 1-1
And $V2$, $\rho 2$, $A2$ are the corresponding values at section 2-2 Then the rate flow at section 1-1 = $\rho 1 A1 V1$
Rate of flow at section 2-2 = $\rho 2\ A2 V2$ According to law of conservation of mass
Rate of flow at section 1-1= Rate of flow at section 2-2



Fluid flowing through a pipe.

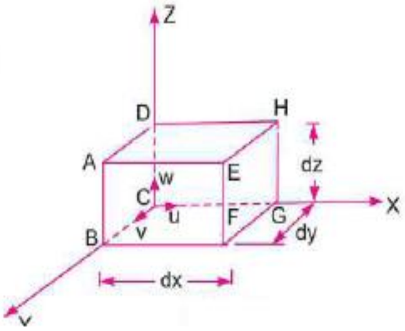
$$\rho 1\ A1\ V1 = \rho 2\ A2\ V2$$

This equation is applicable to the compressible as well as incompressible fluids and is called **"Continuity equation"**. If the fluid is incompressible, then $\rho 1 = \rho 2$ and the continuity equation reduces to

$$A1\ V1 = A2\ V2$$

CONTINUITY EQUATION IN THREE DIMENSIONAL FLOW

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively.



Mass of fluid entering the face ABCD per second

= ρ × velocity in x – direction × Area of ABCD

= ρ × u ×(dy×dz)

Then the mass of fluid leaving the face EFGH per second

$$= \rho \times u \times (dy \times dz) + \frac{\partial}{\partial x} \rho u \, dy \, dz \, dx$$

Gain of mass in x- direction

= Mass through ABCD – Mass through EFGH per second.

$$= \rho u \, dy \, dz - \rho u \, dy \, dz - \frac{\partial}{\partial x} \rho u \, dy \, dz \, dx$$

$$= -\frac{\partial}{\partial x} \rho u \, dy \, dz \, dx$$

$$= -\frac{\partial}{\partial x} \rho u \; dxdydz \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

Similarly the net gain of mass in y- direction.

$$= -\frac{\partial}{\partial y} \rho v \, dx \, dy \, dz \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

In z – direction

$$= -\frac{\partial}{\partial z}(\rho w) dx \, dy \, dz \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

$$\text{Net gain of mass} = -\left[\frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v + \frac{\partial}{\partial z} \rho w\right] dxdydz \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (4)}$$

Since mass is neither created nor destroyed in the fluid element , the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But the mass of fluid in the element is ρ dx dy dz and its rate of increase with time is

$$\frac{\partial}{\partial t}(\rho \; dx. \; dy. \; dz) \text{ or } \frac{\partial \rho}{\partial t} . \, dx \, dy \, dz. \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (5)}$$

Equating the two expressions (4) & (5)

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t} . \, dxdydz$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \text{\_\_\_\_\_\_\_\_ (6)}$$

This equation is applicable to

i) Steady and unsteady flow

ii) Uniform and non- uniform flow , and

iii) Compressible and incompressible flow.

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and hence equation (6) becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

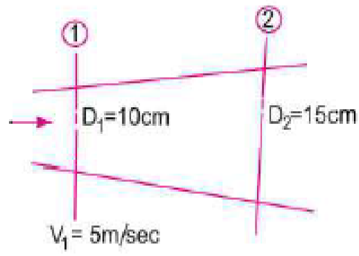
If the fluid is incompressible, then ρ is constant and the above equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This is the continuity equation in three - dimensional flow.

PROBLEM 1. The diameter of a pipe at sections 1 and 2 are 10 cm and 15cms respectively. Find the discharge through pipe, if the velocity of water flowing through the pipe at section 1 is 5m/sec. determine the velocity at section 2.

Given:



At section 1,

D1 = 10cms = 0.1m

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

V1 = 5m/sec

At section 2, D2 = 15cms =0.15m

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

Discharge through pipe Q = A1 ×V1

= 0.007854 ×5 = 0.03927 m3/sec

We have A1 V1 = A2 V2

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = \mathbf{2.22 \text{ m/s. Ans.}}$$

PROBLEM 2. A pipe through which water is flowing is having diameters 20cms and 10cms at cross- sections 1 and 2 respectively. The velocity of water at section 1 is 4 m/sec. Find the velocity head at section 1 and 2 and also rate of discharge?

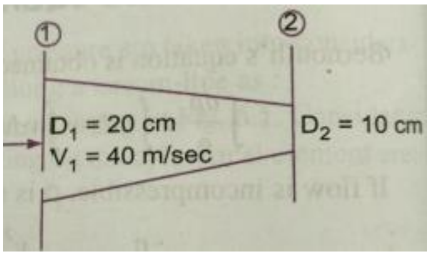
Given: D1 = 20cms = 0.2m

$$A_1 = \frac{\pi}{4} \times 0.2^2 = 0.0314 m^2$$

$V_1 = 4$ m/sec

$D_2 = 10$ cm $= 0.1$ m

$$A_2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 m^2$$



i) Velocity head at section 1 $\frac{V_1^2}{2g} = \frac{4\times4}{2\times9.81} = 0.815m$

ii) Velocity head at section 2 $\frac{V_2^2}{2g}$

To find V_2, apply continuity equation

We have A1 V1   A2 V2

$$V_2 = \frac{A_1V_1}{A_2} = \frac{0.0314\times4}{0.00785} = 16m/sec$$

Velocity head at section 2

$$\frac{V_2^2}{2g} = \frac{16\times16}{2\times9.81} = 13.047m$$

iii) Rate of discharge

$Q = A_1 V_1 = A_2 V_2$

$= 0.0314 \times 4 = 0.1256$ m³/sec

Q = 125.6 Liters/sec