2.4 Transformation of Random Variables:

Let (X, Y) be a continuous two dimensional random variables with JPDF $f_{XY}(x, y)$. Transform X and Y to new random variables $U = h(x, y), V = g(x, y)$.

Then the joint PDF of (U, V) is given by

$$f_{UV}(u, v) = |J|\, f_{XY}(x, y)$$

where $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Procedure to find the Marginal pdf of U & V

(1)Take u as the random variable to which the PDF to be computed and take $v =$ y. (if not given)

(2) Express x and y in terms of u and v.

(3) Find $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

(4) Write the JPDF of (U, V), $f_{UV}(u, v) = |J|\, f_{XY}(x, y)$

(5) Substitute the values of J, x and y.

(6) Find the range of u and v using the range of x and y.

(7) The PDF of U is $f_U(u) = \int_{v=-\infty}^{v=\infty} f_{uv}(u,v)dv$

(8) The PDF of V is $f_V(v) = \int_{u=-\infty}^{u=\infty} f_{uv}(u,v)du$

Problem based on Transformation of Random Variables

1. If the JPDF $f(x,y)$ is given by $f_{XY}(x,y) = x + y; 0 \leq x, y \leq 1$, find PDF of $U = XY$.

Solution:

Given (X, Y) is a continuous $2D$ RV defined in $0 < x < 1$ and $0 < y < 1$.

Also Given $f_{xy}(x,y) = x + y \; 0 \leq x, y \leq 1$

we have to find the PDF of $u = xy$ … … . .. (1)

let $v = y \Rightarrow y = u.$

$(1) \Rightarrow \text{u} = xv \Rightarrow x = \frac{u}{v}$

$\therefore x = \frac{u}{v} \qquad y = v$

$\frac{\partial z}{\partial u} = \frac{1}{v}; \frac{\partial x}{\partial v} = \frac{-u}{v^2}; \frac{\partial y}{\partial u} = 0; \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$J = \frac{1}{v}$$

The JPDF of $(U,V) f_{uv}(u,v) = |J| f_{xy}(x,y)$

$$= \left|\frac{1}{v}\right|(x+y) = \frac{1}{v}\left(\frac{u}{v}+v\right)$$

$$= \frac{u}{v^2}+1$$

$$f_{uv}(u,v) = \frac{u}{v^2}+1$$

To find the range for u and v:

We have $0 \le x \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$

i.e $0 \le u \le v$

Also $0 \le y \le 1 \Rightarrow 0 \le v \le 1$

On combining the two limits, we get $0 \le u \le v \le 1$

$$\therefore f_{uv}(u,v) = \frac{u}{v^2}+1,\ 0 \le u \le v \le 1$$

PDF of U is given by

$f_U(u) = \int_{v=u}^{v=1} f_{uv}(u,v)dv$ $\qquad 0 \le u \le v < 1$

$$= \int_u^1 \left(\frac{u}{v^2}+1\right)dv$$

$$= \int_u^1 (uv^{-2}+1)dv$$

$$= \left[\frac{uv^{-1}}{-1}+v\right]_u^1$$

$= \left(\frac{u}{-1} + 1\right) + 1 - u$

$= -u + 1 + 1 - u$

$= 2 - 2u$

$f_U(u) = 2(1 - u)\ 0 < u < 1$

2. Let $(\boldsymbol{X}, \boldsymbol{Y})$ be a continuous two dimensional randow. with JPDF $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) =$ $\boldsymbol{4xye^{-(x^2+y^2)}} \boldsymbol{x} > \boldsymbol{0}, \boldsymbol{y} > \boldsymbol{0}$. Find the PDF of $\sqrt{\boldsymbol{X^2} + \boldsymbol{Y^2}}$

Solution:

Given (X, Y) is a continuous two dimensional random variables defined in $0 <$ $x < \infty$ and

$0 < \mathrm{y} < \infty$

Given $f(x, y) = 4xye^{-(x^2+y^2)}$, $0 < x < \infty, 0 < y < \infty$

let $u = \sqrt{x^2 + y^2} \ldots\ldots (1)$ Take $v = y \Rightarrow y = v$

(1) => $u^2 = x^2 + y^2$

$u^2 = x^2 + y^2 \qquad y = v$

$x^2 = u^2 - v^2 \Rightarrow$ x=$\sqrt{u^2 - v^2}$

$x\sqrt{u^2 - v^2}$, $y = v$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{u^2 - v^2}}(2u) = \frac{u}{\sqrt{u^2 - v^2}}; \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\frac{1}{\sqrt{u^2 - v^2}}(-2v) = \frac{-v}{\sqrt{u^2 - v^2}}; \frac{\partial y}{\partial v} = 1 = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{u}{\sqrt{u^2-v^2}} & \frac{-v}{\sqrt{u^2-v^2}} \\ 0 & 1 \end{vmatrix}$$

$$J = \frac{u}{\sqrt{u^2-v^2}}$$

PDF of (U, V) is $f_{UV}(u, v) = |J| f_{XY}(x, y)$

$$= \frac{u}{\sqrt{u^2-v^2}} 4xye^{-(x^2+y^2)}$$

$$= \frac{u}{\sqrt{u^2-v^2}} 4\sqrt{u^2 - v^2}(v)e^{-u^2}$$

$$f_{UV}(u, v) = 4uve^{-u^2}$$

To find the range for u and v:

We have $x > 0$ We have $y > 0$

$\sqrt{u^2 - v^2} > 0$ $v > 0$

$u^2 - v^2 > 0$ $\Rightarrow 0 < v < \infty$

$u^2 > v^2 \Rightarrow u > v$

$\Rightarrow v < u$

On combining the two limits, we get $0 < v < u < \infty$

$$f_{UV}(u,v) = 4uve^{-u^2}, 0 < v < u < \infty$$

PDF of *U* is given by

$$f_U(u) = \int_{v=0}^{v=u} f_{uv}(u,v)dv$$

$$= \int_0^u 4uve^{-u^2} dv$$

$$= 4ue^{-u^2} \int_0^u v dv$$

$$= 4ue^{-u^2} \left[\frac{v^2}{2}\right]_0^u$$

$$= 2u^3 e^{-u^2} 0 < u < \infty$$

3. The JPDF to two dimensional random variables X and Y is given by,

$(x,y) = e^{-(x+y)}, x > 0, y > 0$. Find the PDF of $\frac{X+Y}{2}$

Solution:

Given (X,Y) is a continuous two dimensional random variable defined in

$0 < x < \infty$ and

$0 < y < \infty$. Also given $f(x,y) = e^{-(x+y)}; 0 < x < \infty, 0 < y < \infty$

let u $= \frac{x+y}{2}$ ... ... (1). Take $v = y \Rightarrow y = v$

$(1) \Rightarrow u = \frac{1}{2}(x + v)$

$2u = x + vx = 2u - v$

$\therefore x = 2u - v;$

$y = v \frac{\partial x}{\mathrm{d}_u} = 2 \frac{\partial x}{\partial v} = -1; \frac{\partial y}{\partial u} = 0; \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

the PDF of (U, V) is $f_{uv}(u,v) = |J| f_{XY}(x,y)$

$$= 2e^{-(x+y)}$$

$$= 2e^{-(2u-v+v)}$$

$$= 2e^{-2u}$$

To find range for u and v :

We have $x > 0 \Rightarrow 2u - v > 0$

i. $e.$,$2u > v \Rightarrow v < 2u$

Also $y > 0 \Rightarrow v > 0$

$\therefore v < 2u; v > 0$ $\quad 0 < v < 2u < \infty$

On combining the two limits, we get $0 < v < 2u < \infty$

$\therefore f_{UV}(u,v) = 2e^{-2u}, 0 < v < 2u < \infty$

The PDF of *U* is

$$f_U(u) = \int_{v=0}^{v=2u} f_{uv}(u,v)dv$$

$$= \int_0^{2u} 2e^{-2u}dv$$

$$= 2e^{-2u}\int_0^{2u} dv$$

$$= 2e^{-2u}[v]_0^{2u}$$

$$= 2e^{-2u}(2u)$$

$$f_u(u) = 4ue^{-2u}; u > 0$$

UNIT STEP FUNCTION:

$u(x) = 1$ for $x > 0$

$u(x) = 0$ for $x < 0$

1. If X and Y are two independent random variables each normally distributed with mean $= 0$ and variance σ^2 , find the density function of $R = \sqrt{X^2+Y^2}$ and $\phi = \tan^{-1}\left(\frac{Y}{x}\right)$

Solution:

Given that X follows $N(0, \sigma)$.

$$\therefore f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}x^2}; -\infty < x < \infty$$

Also Y follows $N(0, \sigma)$.

$$\therefore f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}y^2}; -\infty < y < \infty$$

Since X and Y are independent, $f_{XY}(x, y) = f_X(x) f_Y(y)$

$$= \frac{1}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}(x^2+y^2)}; -\infty < x < \infty, -\infty < y < \infty$$

We have $r = \sqrt{x^2+y^2}; \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow x = \mathrm{r}\cos\theta, y = \mathrm{r}\sin\theta,$

$\Rightarrow J = r$

JPDF of (R, ϕ) is $f_{R\phi}(r, \theta) = |J| f_{XY}(x, y)$

$$= r\frac{1}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}(x^2+y^2)}$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2}$$

To find the range for r and θ :

We have $-\infty < x < \infty, -\infty < y < \infty$ t.e entire XY plane.

The entire XY plane is transformed into $x = \mathrm{r}\cos\theta, y = \mathrm{r}\sin\theta$

i.e the entire XY plane is transformed into $x^2 + y^2 = r^2$ (a circle of infinite radius)

Whole region is transformed into a circle of infinite radius.

$$\therefore 0 \le r < \infty, 0 \le \theta \le 2\pi$$

$$\therefore f_{R\phi}(r,\theta) = \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} 0 \le r < \infty, 0 \le \theta \le 2\pi$$

The PDF of R is

$$f_R(r) = \int_{r=0}^{\infty} f_{r\theta}(r,\theta)d\theta$$

$$= \int_0^{in} \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} d\theta$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} \int_0^{\infty} d\theta$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2} r^2} [\theta]_0^{2\pi}$$

$$f_R(r) = \frac{r}{\sigma^2} e^{\frac{-1}{2\sigma^2} r^2}; 0 \leq r < \infty$$

The PDF of ϕ is

$$f_\phi(\theta) = \int_{r=0}^{\infty} f_{r\theta}(r,\theta) dr$$

$$= \int_0^{\infty} \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2} r^2} dr$$

$$= \frac{1}{\sigma^2 2\pi} \int_0^{\infty} r e^{\frac{-1}{2\sigma^2} r^2} dr$$

Put $\frac{1}{2\sigma^2} r^2 = t$

$$\frac{1}{2\sigma^2} 2r dr = dt$$

$$r dr = \sigma^2 dt$$

There is no change on the limits

$$f_0(\theta) = \frac{1}{\sigma^2 2\pi} \int_0^{\infty} e^{-t} \sigma^2 dt$$

$$= \frac{1}{2\pi} \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$= \frac{1}{2\pi}(0+1)$

$f_{\phi}(\theta) = \frac{1}{2\pi} 0 \leq \theta \leq 2\pi$

2. The random variables X and Y each follows an exponert distribution with parameter 1 and are independent. Find the PDF of $U = X - 1$

Solution:

Given X and Y follows exponential distribution with parameter with $\lambda = 1$

$\therefore f_x(x) = \lambda e^{-\lambda x}; x > 0$

$= e^{-x}$

$f_y(y) = e^{-y}; y > 0$

Since X and y are independent,

$f_{\mathrm{XY}}(\mathrm{x,y}) = f_x(x) f_y(y)$

$= e^{-x} e^{-y}$

$= e^{-(x+y)}$

let $u = x - y$ ……………..(1) Take $v = y \Rightarrow y = v$

$(1) \Rightarrow u = x\ v \Rightarrow x = u + v$

$x = u + v\ ; y = v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The JPDF of (U, V) is $f_{\text{uv}}(u, v) = |J| f_{XY}(x, y)$

$$= (1)e^{-(x+y)} = e^{-(u+v+v)}$$

$$= e^{-(u+2v)}$$

To find the range for u and v :

We have $x > 0 \Rightarrow u + v > 0 \Rightarrow u > -v$

fie $y > 0 \Rightarrow v > 0$

$\therefore \text{f}_{\text{uv}}(u, v) = e^{-(u+2v)} u > -v, v > 0$

The PDF of U is

$f_u(u) = \int f(u, v) dv$

Since there are two slopes, the region is divided into two sub regions R_1 and R_2

ln R_1:

At $P_1, v = -u$; At Q_1, v

ln R_2 :

At $P_2, v = 0$; At $Q_2, v = \infty$

In R_1 :

$$f_U(u) = \int_{v=-4}^{\infty} f(u,v)dv$$

$$= \int_{-u}^{\infty} e^{-(u+2v)}dv$$

$$= \int_{-u}^{\infty} e^{-u}e^{-2v}dv$$

$$= e^{-u}\int_{-u}^{\infty} e^{-2v}dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}$$

$$= e^{-u}\left[0 - \frac{e^{2u}}{-2}\right]$$

$$= \frac{e^{u}}{2}; u < 0$$

In R_2

$$f_U(u) = \int_{v=0}^{\infty} e^{-u} f(u,v)dv$$

$$= \int_{0}^{\infty} e^{-(u+2v)}dv$$

$$= \int_{0}^{\infty} e^{-u}e^{-2v}dv$$

$$= \int_0^\infty e^{-u}\left[0 - \frac{1}{-2}\right]$$

$$= \frac{e^{-u}}{2}; u > 0$$

$$= e^{-u}\int_0^\infty e^{-2v} dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_0^\infty$$

$$f_U(u) = \begin{cases} \frac{e^{u}}{2} & u < 0 \\ \frac{e^{-u}}{2} & u > 0 \end{cases}$$