Lagrange's Linear Equation

Equations of the form Pp + Qq = R ________ (1), where P, Q and R are functions of x, y, z, are known as Lagrange solve this equation, let us consider the equations u = a and v = b, where a, b are arbitrary constants and u, v are functions of x, y, z.

Note :

To solve the Lagrange's equation,we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping or by the method of multipliers.

PROBLEMS UNDER METHODS OF MULTIPLIERS

Solve $(\boldsymbol{mz - ny})\boldsymbol{p} + (\boldsymbol{nx - lz})\boldsymbol{q} = \boldsymbol{ly - mx}$

The given PDE is a Lagrange's linear equation with

$$p = mz - ny, Q = nx - lz, R = ly - mx.$$

subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \dots \dots \dots$$

Using the multipliers (x, y, z), each of the ratios in (1) is equal to

$$\frac{xdx+ydy+zdz}{x(nz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{xmz-xny+ynx-lyz+yzl-xzm}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating, we get

$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$

$x^2+y^2+z^2=c_1$

$$u=x^2+y^2+z^2$$

Using the multipliers (l,m,n), each of the ratios in (1) is equal to

$$\frac{ldx+mdy+ndz}{l(mx-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{ldx+mdy+ndz}{lmz-nly+mnx-lmz+nly-mnx}$$

$$=\frac{ldx+mdy+ndz}{0}$$

$\therefore ldx+mdy+ndz=0$

Integrating, we get

$lx+my+nz=c_2$

$$v=lx+my+nz$$

The general solution of the given equation is $f(u,v)=0$

$$f(x^2+y^2+z^2,lx+my+nz)=0$$

Solve $(3z-4y)\frac{\partial z}{\partial x}+(4x-2z)\frac{\partial z}{\partial y}=2y-3x$.

Sol: The given PDE is a Lagrange's linear equation with

$$P=3z-4y,Q=4x-2z,R=2y-3x$$

The subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Using the multipliers (2,3,4), each of the ratios in (1) is equal to

$$\frac{2dx+3dy+4dz}{2(3z-4y)+3(4x-2z)+4(2y-3x)}=\frac{2dx+3dy+4dz}{6z-8y+12x-6z+8y-12x}$$

$$=\frac{2dx+3dy+4dz}{0}$$

$\therefore 2dx+3dy+4dz=0$

Integrating, we get

$2x+3y+4z=c_1$

$$u=2x+3y+4z$$

Using the multipliers (x,y,z), each of the multipliers in (1) is equal to

$$\frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)} = \frac{xdx+ydy+zdz}{3xz-4xy+4xy-2yz+2yz-3xz}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$x^2 + y^2 + z^2 = c_2$$

$$v = x^2 + y^2 + z^2$$

The general solution of the given equation is $f(u, v) = 0$

$$f(2x + 3y + 4z, x^2 + y^2 + z^2) = 0$$

Solve $x(y - z)p + y(z - x)q = z(x - y)$.

Sol: The given PDE is a Lagrange's linear equation with

$$P = x(y - z), Q = y(z - x), R = z(x - y)$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \dots\dots\dots (1)$$

Using the multipliers (1,1,1),each of the ratios in (1) is equal to

$$\frac{dx+dy}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{xy-xz+yz-xy+xz-zy}\ \frac{dx+dy+dz}{0}$$

$$\therefore dx + dy + dz \quad = 0$$

Integrating, we get

$$x + y + z = c_1$$

$$u \quad = x + y + z = c_1$$

$$Using\ the\ multipliers \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right), eachoftheratiosin(1) is\ equal\ to$$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}x(y-z)+\frac{1}{y}y(z-x)+\frac{1}{z}z(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz \quad = 0$$

Integrating, we get

$$\log x + \log y + \log z \quad = \log c_2$$
$$\log(xyz) \quad = \log c_2$$
$$xyz \quad = c_2$$
$$v \quad = xyz$$

The general solution of the given equation is $f(u, v) = 0$

$$f(x + y + z, xyz) = 0$$

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Sol: The given PDE is a Lagrange's linear equation with

$$P = x(y^2 - z^2), Q = y(z^2 - x^2), R = z(z^2 - y^2)$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \dots\dots\dots (1)$$

Using the multipliers (x, y, z), each of the multipliers in (1) i s equal to

$$\frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2)} + z^2(x^2 - y^2)$$

$$= \frac{xdx + ydy + zdz}{x^2y^2 - x^2z^2 + y^2z^2 - x^2y^2 + x^2z^2 - y^2z^2}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \quad = \frac{c_1}{2}$$
$$x^2 + y^2 + z^2 \quad = c_1$$
$$u \quad = x^2 + y^2 + z^2$$

Using the multipliers $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$, each of the ratio in (1) is equal to

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz \quad = 0$$

Integrating, we get

$$\begin{aligned} \log x+\log y+\log z &= \log c_2 \\ \log(xyz) &= \log c_2 \\ xyz &= c_2 \\ v &= xyz \end{aligned}$$

The general solution of the given equation is $f(u,v)=0$

$$f(x^2+y^2+z^2,\text{xyz})=0$$

Solve $x^2(\mathbf{y}-\mathbf{z})\mathbf{p}+y^2(\mathbf{z}-\mathbf{x})\mathbf{q}=z^2(\mathbf{x}-\mathbf{y})$

Sol: the given PDE is a Lagrange's linear equation with

P = $x^2(\text{y}-\text{z}),\text{Q}=y^2(\text{z}-\text{x}),\text{R}=z^2(\text{x}-\text{y})$

The subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{x^2(\text{y}-\text{z})}=\frac{dy}{y^2(\text{z}-\text{x})}=\frac{dz}{z^2(\text{x}-y)} \dots\dots\dots \ (1)$$

Using the multipliers $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$, each of the ratio in (1) is equal to

$$\begin{aligned} \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz \quad &= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{xy-xz+yz-xy+xz-yz} \\ (y-z)+y(z-x)+z(x-y) \quad &= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0} \\ \therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz \quad &= 0 \end{aligned}$$

Interating, we get

$$\begin{aligned} \log x + \log y + \log z &= \log c_1 \\ \log(xyz) &= \log c_1 \\ xyz &= c_1 \\ u &= xyz \end{aligned}$$

Wing the multipliers $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$, each of the ratio in (1) is equal to

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get

$$\begin{aligned} \log x + \log y + \log z &= \log c_1 \\ \log(xyz) &= \log c_1 \\ xyz &= c_1 \\ u &= xyz \end{aligned}$$

Using the multipliers $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$, each of the ratio in (1) is equal to

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get

$$\begin{aligned} -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} &= -c_2 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= c_2 \\ v &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \end{aligned}$$

The general solution of the given equation is $f(u, v) = 0$

$$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Sol: The given PDE is a Lagrange's linear equation with

$$P = x^2 - yz, Q = y^2 - xz, R = z^2 - xy$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - xz} = \frac{dz}{z^2 - xy} \dots\dots\dots (1)$$

Using the multipliers $(y + z, z + x, x + y)$, each of the ratio in (1) is equal to

$$\frac{(y+z)dx + (z+x)dy + (x+y)dz}{(y+z)(x^2 - yz) + (z+x)(y^2 - xz) + (x+y)(z^2 - xy)}$$

$$= \frac{(y+z)dx + (z+x)dy + (x+y)dz}{x^2y - y^2z + x^2z - yz^2 + zy^2 - xz^2 + xy^2 - x^2z + xz^2 - x^2y + yz^2 - xy^2}$$

$$= \frac{(y+z)dx + (z+x)dy + (x+y)dz}{0}$$

$$(y+z)dx + (z+x)dy + (x+y)dz = 0$$

$$ydx + zdx + zdy + xdy + xdz + ydz = 0$$

$$ydx + xdy + zdx + xdz + zdy + ydz = 0$$

$$d(yx) + d(zx) + d(zy) = 0$$

Integrating, we get

$$yx + zx + zy = c_1$$

$$u = xy + +zx + zy$$

Each of the ratios in (1) is equal to

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - xz)} = \frac{dy - dz}{(y^2 - xz) - (z^2 - xy)}$$

$$\frac{d(x-y)}{x^2 - yz - y^2 + xz} = \frac{d(y-z)}{y^2 - xz - z^2 + xy}$$

$$\frac{d(x-y)}{x^2 - y^2 + xz - yz} = \frac{d(y-z)}{y^2 - z^2 + xy - zx}$$

$$\frac{d(x-y)}{(x+y)(x-y) + z(x-y)} = \frac{d(y-z)}{(y+z)(y-z) + x(y-z)}$$

$$\frac{d(x-y)}{(x-y)[x+y+z]} = \frac{d(y-z)}{(y-z)[y+z+x]}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get

$$\begin{aligned}
\log(x-y) &= \log(y-z) + \log c_2 \\
\log(x-y) - \log(y-z) &= \log c_2 \\
\log\left(\frac{x-y}{y-z}\right) &= \log c_2 \\
\frac{x-y}{y-z} &= c_2 \\
v &= \frac{x-y}{y-z}
\end{aligned}$$

The general solution of the given equation is $f(u, v) = 0$

$$f\left(xy + yz + zx, \frac{x-y}{y-z}\right) = 0$$

PROBLEMS UNDER GROUPING

Solve $xp + yq = z$

Given $xp + yq = z$

PDE is a Lagrange's linear equation with

$$P = x, Q = y, R = z$$

diary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

the first two ratios, we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x = \log y + \log c_1$$

$\log x - \log y = \log c_1$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1$$

$u = \frac{x}{y}$

Taking the last two ratios, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get

$$\begin{aligned} \log y &= \log \dot{z} + \log c_2 \\ \log y - \log z &= \log c_2 \\ \log\left(\frac{y}{z}\right) &= \log c_2 \\ \frac{y}{z} &= c_2 \\ v &= \frac{y}{z} \end{aligned}$$

The general solution of the given equation is $f(u, v) = 0$

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Solve $xp + yq = x$.

Sol: The given PDE is a Lagrange's linear equation with

$P = x, Q = y, R = x$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$

Taking the first two ratios, we get

$\frac{dx}{x} = \frac{dy}{y}$

Integrating we get

$$\begin{aligned} \log x &= \log y + \log c_1 \\ \log x - \log y &= \log c_1 \\ \log\left(\frac{x}{y}\right) &= \log c_1 \\ \frac{x}{y} &= c_1 \\ u &= \frac{x}{y} \end{aligned}$$

Taking the first and the third ratios, we get

$$\frac{dx}{x} = \frac{dz}{x}$$
$$dx = dz$$

Integrating we get

$$\begin{aligned} x &= z + c_2 \\ c_2 &= x - z \\ v &= x - z \end{aligned}$$

The general solution of the given equation is $f(u, v) = 0$

$$f\left(\frac{x}{y}, x - z\right) = 0$$

Solve $p\tan x + q\tan y = \tan z$.

Sol: Given $p \tan x + q\tan y = \tan z$

The given PDE is a Lagrange's linear equation with

$$P = \tan x, Q = \tan y, R = \tan z$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two ratios, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x dx = \cot y dy$$

Integrating, we get

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$= \log(\sin y c_1)$$

$$\sin x = \sin y c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$$u = \frac{\sin x}{\sin y}$$

Taking the last two ratios, we get

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\cot y dy = \cot z dz$$

Integrating, we get

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$= \log(\sin z c_2)$$

$$\sin y = \sin z c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$$v = \frac{\sin y}{\sin z}$$

The general solution of the given equation is $f(u, v) = 0$

$$\text{i.e., } f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$