2.6. DISCRETE FOURIER TRANSFORM

In mathematics, the discrete Fourier transform (DFT) converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT), which is a complex-valued function of frequency.

The discrete time Fourier Transform for a discrete time a periodic signal x (n) was determined to be given by

$X(\omega) = X(z)\,|_{z=e^{j\omega}}$

Where

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Symbolically the Fourier transform of x(n) is defined as, F[x(n)]

Where, F is the operator that represents Fourier transform

$X(e^{j\omega}) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

The Fourier transform of a signal is said to exist if it can be expressed in a valid functional form. Since the computation of Fourier transform involves summing infinite number of terms, the Fourier transform exists only for the signals that are absolutely summable, given signal or signal spectrum.

$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

2.6.1. MAGNITUDE AND PHASE REPRESENTATION:

The Fourier transform $X(e^{j\omega})$ of a signal x(n) represents the frequency content of x(n).We can say that by taking the Fourier transform, the signal x(n) is decomposed into its frequency components. Hence $X(e^{j\omega})$ is also called frequency spectrum of discrete time signal or signal spectrum.

The $X(e^{j\omega})$ is a complex valued function of ω, and so it can be expressed in rectangular form as

$X(e^{j\omega}) = X_r(e^{j\omega}) + j\,X_i(e^{j\omega})$

Where, $X_r(e^{j\omega})$ =Real part of $X(e^{j\omega})$

$X_i(e^{j\omega})$ = Imaginary part of $X(e^{j\omega})$

The polar form of $X(e^{j\omega})$ is,

$X(e^{j\omega}) = |X(ej\omega)|\,\angle X(e^{j\omega})$

Where, $|X(ej\omega)|$ = Magnitude spectrum

$\angle X(e^{j\omega})$ = Phase spectrum

The magnitude spectrum is defined as

$|X(e^{j\omega})|^2 = X(e^{j\omega})\, X^*(e^{j\omega})$ (or) $|X(e^{j\omega})| = \sqrt{X(ej\omega)\, X^*(ej\omega)}$

Where, $X^*(e^{j\omega})$ is complex conjugate of $X(e^{j\omega})$

The phase spectrum is defined as,

$\angle X(e^{j\omega}) = \text{Arg}\,[X(e^{j\omega})] = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]$

2.6.2. PROPERTIES OF DISCRETE FOURIER TRANSFORM

Linearity Property

The linearity property of DTFT says that the operation obeys the principle of superposition.

$x[n]=ax_1[n]+bx_2[n] \leftrightarrow aX_1(e^{j\omega}) = aX_1(e^{j\omega}) + b\,X_2(e^{j\omega})$

Time-Delay Property

When we first studied sinusoids, the phase was shown to depend on the time-shift of the signal. The simple relationship was “phase equals the negative of frequency times time-shift.” This concept carries over to the general case of the Fourier transform. The time-delay property of the DTFT states that time-shifting results in a phase change in the frequency domain

$y[n] = x[n-n_d] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})\, e^{-j\omega n_d}$

Frequency-Shift Property

Consider a sequence $y[n] = e^{j\hat{\omega}_c n}\, x[n]$ where the DTFT of x[n] is $X(e^{j\hat{\omega}})$. The multiplication by complex exponential causes a frequency shift in the DTFT of y[n] compared to the DTFT of x[n]. By definition, the DTFT of y[n] is

$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_c} n\, x[n] e^{-j\omega n}$

If we combine the exponentials in the summation on the right side of eqn we obtain

$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_c)n}$

$=X(e^{j(\omega-\omega_c})$

Therefore, we have proved the following general property of the DTFT:

$y(n)=e^{j\omega cn}\, x[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j(\omega-\omega c)})$

Convolution

Perhaps the most important property of the DTFT concerns the DTFT of a sequence that is the discrete-time convolution of two sequences. The following property says that the DTFT transforms convolution into multiplication.

y(n)=x[n]*h[n] $\leftrightarrow Y\left(e^{j\omega}\right)H(e^{j\omega})$