Definition

Let V be a finite dimensional inner product space and let T be a linear operator on V. Then there exist a unique function $T^*: V \to V$ such that $\langle T(x), y\rangle$, $\langle x, T^*(y)\rangle$ for all $x, y \in V$. The linear operator T^* is called adjoint of operator T.

Theorem 3.14: Let T be a linear functional on a finite dimensional inner product space V. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for every $x \in V$.

Proof: Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of V.

Let $y = \overline{g(v_1)}v_1 + \overline{g(v_2)}v_2 + \cdots + \overline{g(v_n)}v_n$

Define $h: V \to F$ by $h(x) = \langle x, y\rangle$ for every $y \in V$.

It is clear that h is linear.

Then for $i = 1,2, \dots \dots, n$,

$h(v_i) = \langle v_i, y\rangle = \left\langle v_i, \overline{g(v_1)}v_1 + \overline{g(v_2)}v_2 + \cdots + \overline{g(v_n)}v_n\right)$

$= \left\langle v_1, \overline{g(v_1)}v_i\right\rangle \left[\because \left\langle v_i, v_j\right\rangle = 0 \text{ for } i \neq j\right]$

$= g(v_i)\langle v_i, v_i\rangle = g(v_i)\|v_i\|^2 \left[\because \|v_i\|^2 = 1\right] \therefore h(v_i) = g(v_i)$

This is true for each $v_i, i = 1,2, \dots, n$

$$\therefore h = g$$

We have to prove the uniqueness.

Now suppose that y' is another vector in V for which

$$g(x) = \langle x, y'\rangle \text{ for each } x \in V$$

Then

$$\langle x, y\rangle = \langle x, y'\rangle$$

$$\Rightarrow (x, y) - \langle x, y'\rangle = 0 \Rightarrow (x, y - y') = 0 \Rightarrow y - y' = 0 \Rightarrow y = y'$$

$$\therefore \text{y is unique}$$

Let T be a linear operator on a finite dimensional inner prods then there exists a unique linear operator T' on V such that

$$\langle \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{T}^*(\boldsymbol{y}) \rangle \textbf{ for every } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}.$$

Proof: Let y be an arbitrary but fixed element of V.

g: V $\rightarrow= F$ by $g(x) = \langle T(x), y \rangle$ for every $y \in V$.

First we prove that g is linear.

La $x_1, x_2 \in V$ and $\alpha \in F$.

$$\begin{aligned} (i) g(x_1 + x_2) &= (T(x_1 + x_2), y) \\ &= \langle T(x_1) + T(x_2), y \rangle [\because T \text{ is linear }] \\ &= \langle T(x_1), y \rangle + \langle T(x_2), y \rangle \\ &= g(x_1) + g(x_2) \end{aligned}$$

$$\begin{aligned} (ii) g(\alpha x_1) = &\quad \langle T(\alpha x_1), y \rangle \\ = &\quad \langle \alpha T(x_1), y \rangle [\because T \text{ is linear }] \\ = &\quad \alpha \langle T(x_1), y \rangle \\ = &\quad \alpha g(x_1) \end{aligned}$$

Therefore g is a linear transformation on V.

By Theorem 3.14, There exists a unique vector $y' \in V$ such that

Define $T^*: V \rightarrow V$ by $T^*(y) = y'$ for $y \in V$.

Therefore $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for each $x \in V$.

We have to prove that T^* is linear

Let $y_1, y_2 \in V$ and $\alpha \in F$.

(i)
$$\begin{aligned} \langle x, T^*(y_1 + y_2) \rangle &= \langle T(x), y_1 + y_2 \rangle \\ &= \langle T(x), y_1 \rangle + \langle T(x), y_2 \rangle \\ &= \langle x, T^*(y_1) \rangle + \langle x, T^*(y_2) \rangle \end{aligned}$$

Since x is arbitrary,

$$\langle T^*(y_1 + y_2) = T^*(y_1) + T^*(y_2)$$

(ii) $\langle x, T^*(\alpha y_1) \rangle == \langle T(x), \alpha y_1 \rangle$

$$\begin{aligned} &= \bar{\alpha} \langle T(x), y_1 \rangle \\ &= \bar{\alpha} \langle x, T^*(y_1) \rangle \\ &= \langle x, \alpha T^*(y_1) \rangle \end{aligned}$$

Since x is arbitrary,

$$T^*(\alpha y_1) = \alpha T^*(y_1)$$

Therefore T^* is linear.

Finally, we need to show that T^* is unique. Suppose that $U: V \to V$.

is linear and that it satisfies $\langle T(x), y\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$. Then

$\langle x, T^*(y)\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$, so

$$T^* = U.$$

Theorem 3.16: Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then $[T^*]_\beta = [T]_\beta^*$

Proof: Let $A = [T^*]_\beta$ and $B = [T]_\beta^*$ and, $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V. Then

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i\rangle \\ &= \overline{\langle v_l, T^*(v_j)\rangle} \\ &= \overline{\langle T(v_l), (v_j)\rangle} \\ &= \bar{A}_{ji} \\ &= A_{ij}^* \end{aligned}$$

Thus $B = A^*$

Theorem 3.17: Let T and U be linear operators on a finite dimensional inner product space V and $\alpha \epsilon F$. Then

(i) $(T + U)^* = T^* + U^*$

(ii) $(\alpha T)^* = \bar{\alpha} T^*$

(iii) $(TU)^* = U^* T^*$

(iv) $(T^*)^* = T$

(v) $I^* = I$

Proof

(i) Let $x, y \in V$

$$\begin{aligned} \langle (T+U)x, y\rangle = \quad & \langle T(x) + U(x), y\rangle \\ &= \langle T(x), y\rangle + \langle U(x), y\rangle \\ &= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle \end{aligned}$$

$$= \langle x, T^*(y) + U^*(y)\rangle$$
$$= \langle x, (T^* + U^*)y\rangle$$
$$\therefore \langle (T+U)x, y\rangle = \langle x, (T^* + U^*)y\rangle$$
$$\Rightarrow \langle x, (T+U)^* y\rangle = \langle x, (T^* + U^*)y\rangle$$

By the uniqueness of adjoint implies

$$(T+U)^* = T^* + U^*$$

(ii) Let $\alpha \in F$ and $x, y \in V$

$$\langle (\alpha T)(x), y\rangle = \langle \alpha T(x), y\rangle$$
$$= \alpha\langle T(x) y\rangle$$
$$= \alpha\langle x, T^*(y)\rangle$$
$$\langle (\alpha T)x, y\rangle = \langle x, \bar{\alpha} T^*(y)\rangle$$
$$\therefore \langle x, (\alpha T)^* y\rangle = \langle x, \bar{\alpha} T^*(y)\rangle$$

By the uniqueness of the adjoint implies

$$(\alpha T)^* = \bar{\alpha} T^*$$

(iii) Let $x, y \in V$

$$\langle (TU)(x), y\rangle = \langle T(U(x)), y\rangle$$
$$= \langle U(x), T^*(y)\rangle$$
$$\langle (TU)(x), y\rangle = \langle x, U^*(T^*(y))\rangle$$
$$= \langle x, (U^*T^*)(y)\rangle$$
$$\therefore \langle (TU)(x), y\rangle = \langle x, (U^*T^*)(y)\rangle$$
$$\langle x, (TU)^* y\rangle = \langle x, (U^*T^*)(y)\rangle$$

By the uniqueness the adjoint implies

$$(TU)^* = U^*T^*$$

(iv) Let $x, y \in V$

$$\langle T^*(x), y\rangle = \overline{\langle y, T^*(x)\rangle}$$
$$= \overline{\langle T(y), x\rangle}$$
$$= \langle x, T(y)\rangle$$
$$\therefore (T^*(x), y) = \langle x, T(y)\rangle$$
$$\langle x, (T^*)^*(y)\rangle = \langle x, T(y)\rangle$$

By uniqueness of adjoint implies

$$(T^*)^* = \mathrm{T}$$

(v|) Let $x, y \in V$

$$\langle \mathrm{I}x, y\rangle = \langle x, y\rangle$$

$$= \langle x, \mathrm{I}y\rangle (\because \mathrm{I}(\mathrm{y}) = \mathrm{y})$$

$$\Rightarrow \langle x, \mathrm{I} * (y)\rangle = \langle x, \mathrm{I}y\rangle$$

By uniqueness of adjoint implies

$$\mathrm{I}^* = \mathrm{I}$$