Contour Integration

Evaluation of Real Integrals

The evaluation of certain types of real definite integrals of complex functions over suitable closed paths or contours and applying Cauchy's Residue theorem is known as Contour Integration.

Type 1: Integration round the unit circle

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is a rational function in $cos\theta$ and $\sin\theta$

To evaluate this type of integrals

We take the unit circle $|z| = 1$ as the contour C.

On $|z| = 1, let\ z = e^{i\theta}$

$$\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\therefore d\theta = \frac{dz}{iz}$$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

and, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$

$|z| = 1 \Rightarrow \theta$ varies from $0\ to\ 2\pi$

$$\therefore \int_0^{2\pi} f\ (\cos\theta, \sin\theta)d\theta = \int_c\ f\ \left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Example: Evaluate $\int_o^{2\pi} \frac{d\theta}{5+4\sin\theta}$ using Contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and\ \sin\theta = \frac{z^2-1}{2iz}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{\frac{dz}{iz}}{5+4\left(\frac{z^2-1}{2iz}\right)}\ where\ c\ is\ |z| = 1$$

$$= \int_c \frac{\frac{dz}{iz}}{\frac{5iz+2z^2-2}{iz}}$$

$$= \int_c \frac{dz}{2z^2+5iz-2}$$

$$= \int_c f(z)\,dz \quad \ldots (1)$$

Where, $f(z) = \frac{1}{2z^2+5iz-2}$

To Evaluate, $\int_c f(z)\, dz$

To find poles of $f(z)$, put $2z^2 + 5iz - 2 = 0$

$$z = \frac{-5i \pm \sqrt{-25+16}}{4} = \frac{-5i \pm 3i}{4}$$

$z = -\frac{i}{2}, -2i$ are poles of order one

Given $C\ is\ |z| = 1$

Consider $z = -\frac{i}{2}$

$\Rightarrow |z| = \left|\frac{-i}{2}\right| = \frac{1}{2} < 1$

$\therefore z = -\frac{i}{2}$ lies inside C

Consider $z = -2i$

$\Rightarrow |z| = |-2i| = 2 > 1$

$\therefore z = -2i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{2}$

$$[Res\ f(z)]_{z=-\frac{i}{2}} = \lim_{z\to -\frac{i}{2}} \left(z + \frac{i}{2}\right) f(z)$$

$$= \lim_{z\to -\frac{i}{2}} \left(z + \frac{i}{2}\right) \frac{1}{2\left(Z+\frac{i}{2}\right)(Z+2i)}$$

$$= \frac{1}{2\left(-\frac{i}{2}+2i\right)} = \frac{1}{3i}$$

$\therefore$ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i \quad [Sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

Example: Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$\Rightarrow d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2iz}$

$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_c \frac{dz/iz}{13+5\left(\frac{z^2-1}{2iz}\right)}$ $where\ Cis\ |z| = 1$

$$= \int_c \frac{dz/iz}{\frac{26iz+5z^2-5}{2iz}}$$

$$= 2\int_c \frac{dz}{5z^2+26iz-5}$$

$$= 2\int_c f(z)dz \quad \ldots(1)$$

Where, $f(z) = \frac{1}{5z^2+26iz-5}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z)$, put $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm 24i}{10}$$

$$\Rightarrow z = -\frac{i}{5}, -5i \text{ are poles of order one.}$$

Given C is $|z| = 1$

Consider $z = -\frac{i}{5}$

$$\Rightarrow |z| = \left|-\frac{i}{5}\right| = \frac{1}{5} < 1$$

$\therefore z = -\frac{i}{5}$ lies inside C

Consider $z = -5i$

$$\Rightarrow |z| = |-5i| = 5 > 1$$

$\therefore z = -5i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{5}$

$$[Res\, f(z)]_{z=-\frac{i}{5}} = \lim_{z\to -\frac{i}{5}} \left(z + \frac{i}{5}\right) f(z)$$

$$= \lim_{z\to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{5z^2+26iz-5}$$

$$= \lim_{z\to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{(5z+i)(z+5i)}$$

$$= \lim_{z\to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{(5(z+\frac{i}{5})(z+5i)}$$

$$= \frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

$\therefore$ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[Sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}$$

Example: Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a > b > 0$ byusing contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_c \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} \text{ where } c \text{ is } |z| = 1$$

$$= \int_c \frac{dz/iz}{\frac{2az+bz^2+b}{2z}}$$

$$= \frac{2}{i}\int_c \frac{dz}{bz^2+2az+b}$$

$$= \frac{2}{i}\int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{bz^2+2az+b}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z), put\ bz^2 + 2az + b$

$$z = \frac{-2a \pm \sqrt{4(a^2-b^2)}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a+\sqrt{a^2-b^2}}{b}$ are poles of order one.

Clearly, $z = \frac{-a+\sqrt{a^2-b^2}}{b} = \propto$ lies inside c

$and\ z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside c

Since a > b, we can write $bz^2 + 2az + b = b(z-\alpha)(z-\beta)$

Find the residue for inside pole $z = \alpha$

$$Res\ f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy residue theorem

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]$$

$$= \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$(1) \Longrightarrow \int_o^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\left[\frac{\pi i}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example: Evaluate $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \ and \cos\theta = \frac{z^2+1}{2z}$$

$$\cos 3\theta = Real\ part\ of\ e^{i3\theta} = R.P\ (z^3)$$

$$\therefore \int_O^{2\pi} \frac{\cos 3\theta\ d\theta}{5-4\cos\theta} = \int_O^{2\pi} \frac{R.P\frac{(z^3)dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)} \ where\ C\ is\ |z| = 1$$

$$= R.P\int_c \frac{z^3 dz/iz}{\frac{5z-(2z^2+2)}{z}}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_c f(z)\, dz \qquad \dots(1)$$

Where, $f(z) = \frac{z^3}{2z^2-5z+2}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z), put\ 2z^2 - 5z + 2 = 0$

$$z = \frac{5+\sqrt{25-16}}{4} = \frac{5\pm3}{4}$$

$\Rightarrow z = 2, \frac{1}{2}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = \frac{1}{2}$

$$\Rightarrow |z| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = \frac{1}{2}$ lies inside C

Consider $z = 2$

$$\Rightarrow |z| = |2| = 2 > 1$$

$\therefore z = 2$ lies outside C

Find the residue for inside pole $z = \frac{1}{2}$

$$[Res\ f(z)]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{(2z-1)(z-2)}$$

$$= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = R.P\,\left(-\frac{1}{i}\right)\left(-\frac{\pi i}{12}\right) = \frac{\pi}{12}$$

Example: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{10-6\cos\theta}$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and\ \cos\theta = \frac{z^2+1}{2z}$$

$$\cos 2\theta = Real\ part\ of\ e^{i2\theta} = R.P\,(Z^2)$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\,d\theta = \int_0 \frac{1-R.P\frac{(z^2)dz}{iz}}{10-6\left(\frac{z^2+1}{2z}\right)}$$

$where\ \ C\ is\ |z| = 1$

$$= R.P\,\int_C \frac{(1-z^2)dz/iz}{\frac{10z-3z^2-3}{z}}$$

$$= R.P\,\left(-\frac{1}{i}\right)\int_C \frac{(1-z^2)dz}{3z^2-10Z+3}$$

$$= R.P\,\left(-\frac{1}{i}\right)\int_C f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{1-z^2}{3z^2-10z+3}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z), put\ 3z^2 - 10z + 3 = 0$

$$z = \frac{10\pm\sqrt{100-36}}{6} = \frac{10\pm 8}{6}$$

$\therefore z = 3, \frac{1}{3}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = \frac{1}{3}$

$$\Rightarrow |z| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$$

$\therefore z = \frac{1}{3}$ lies inside C

Consider $z = 3$

$$\Rightarrow |z| = |3| = 3 < 1$$

$\therefore z = 3$ lies outside C

Find the residue for inside pole $z = \frac{1}{3}$

$$[Res\, f(z)]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)f(z)$$

$$= \lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)\frac{(1-Z^2)}{3\left(Z-\frac{1}{3}\right)(Z-3)}$$

$$= \frac{1-\left(\frac{1}{3}\right)^2}{3\left(\frac{1}{3}-3\right)} = -\frac{1}{9}$$

∴ By Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(-\frac{1}{9}\right)$$

$$(1) \Rightarrow \int_0^{2\pi}\frac{sin^2\theta\, d\theta}{5-3\cos\theta} = R.P\left(-\frac{1}{i}\right)\left(-\frac{2\pi i}{9}\right) = \frac{2\pi}{9}$$

Example: Using Contour Integration, evaluate the real integral $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{Z^2+1}{2Z}$$

Now, $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, if\ f(2a-x) = f(x)\right]$$

$$\therefore \frac{1}{2}\int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta = \frac{1}{2}\int_C \frac{1+\left[1+2\left(\frac{(z^2+1)}{2z}\right)\right]\frac{dz}{iz}}{5+4\left(\frac{(z^2+1)}{2z}\right)}$$

$$= \frac{1}{2i}\int_C \frac{(z^2+z+1)}{z(2z^2+5z+2}$$

$$= \frac{1}{2i}\int_C f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{z^2+z+1}{Z(2z^2+5z+2)}$

To evaluate $\int_C f(z)dz$

To find poles of $f(z), put\ z(2z^2+5z+2) = 0$

$$\Rightarrow z = 0;\ 2z^2+5z+2 = 0$$

$\Rightarrow z = 0; z = -2, z = -\frac{1}{2}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = 0$

$$\Rightarrow |z| = |0| = 0 < 1$$

$\therefore z = 0$ lies inside C

Consider $z = -2$

$$\Rightarrow |z| = |-2| = 2 > 1$$

$\therefore z = -2$ lies outside C

Consider $z = -\frac{1}{2}$

$$\Rightarrow |z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = -\frac{1}{2}$ lies inside C

Find the residue for the inside pole

(i) When $z = 0$

$$[Res\, f(z)]_{z=0} = \operatorname*{Lt}_{z\to 0} (z-0) f(z)$$

$$\operatorname*{Lt}_{z\to 0} z \, \frac{(z^2+z+1)}{Z(2Z^2+5z+2)} = \frac{1}{2}$$

(ii) When $z = -\frac{1}{2}$

$$[Res\, f(z)_{z=-\frac{1}{2}} = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z)$$

$$= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{z^2+z+1}{Z(2z+1)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{z^2+z+1}{Z\,2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{\frac{1}{4}-\frac{1}{2}+1}{2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right)}$$

$$= \frac{\frac{3}{4}}{-\frac{3}{2}} = -\frac{1}{2}$$

$\therefore$ By Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i\, [Sum\ of\ residues]$$

$$= 2\pi i \left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

$$(1) \Rightarrow \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{2i}[0] = 0$$

Type II: Integration around semi – circular contour

Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx,$

where $f(x)$ and $g(x)$ are polynomials in x, such that the degree of $f(x)$ is less than that of $g(x)$ atleast by two and $g(x)$ does not vanish for any value of x.

Let C be a closed contour of real axis from $-R$ to R and semicircle $'S'$ of radius R above real axis.

Thus,

$$\int_C \frac{f(z)}{g(z)} = \int_{-R}^{R} \frac{f(x)}{g(x)} dx + \int_S \frac{f(z)}{g(z)} dz$$

As $R \to \infty, \int_C \frac{f(z)}{g(z)} dz \to 0$ by Cauchy's lemma

$$= \int_C \frac{f(z)}{g(z)} dz = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ where $a > b > 0$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int_C \frac{z^2 dz)}{(z^2+a^2)(z^2+b^2)} where$$

Where C is the upper semi circle

$$= \int_C f(z)dz \quad \dots (1)$$

Where, $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, z = \pm bi,$ are poles of order one.

Here $z = ai, bi$ lies in upper, half of the z −plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\ f(z)]_{z\to ai} = \lim_{z\to ai} (z - ai) \frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai\ (b^2-a^2)}$$

$$= \frac{a}{2i\ (a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\ f(z)]_{z=bi} = \lim_{z\to bi} (z - bi) f(z)$$

$$= \lim_{z\to bi} (z - bi) \frac{z^2}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= -\frac{b^2}{(a^2-b^2)2bi}$$

$$= -\frac{b}{2i\ (a^2-b^2)}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i\ (a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{a-b}{(a-b)(a+b)}\right]$$

$$= \frac{\pi}{a+b}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example: Evaluate $\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a > 0, b > 0$

Solution:

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(z^2+a^2)(z^2+b^2)}$$

Replacement put $x = z$

$$\Rightarrow dx = dz$$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Where C is the upper semi circle

$$= \frac{1}{2}\int_{c} f(z)dz \ldots (1)$$

Where, $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one.

Here $z = ai, bi$ lies in the upper half of the z- plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai}(z - ai)f(z)$$

$$= \lim_{z\to ai}(z - ai)\frac{1}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{1}{2ai(b^2-a^2)} = -\frac{1}{2ai\,(a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\, f(z)]_{z=bi} = \lim_{z\to ai}(z - bi)f(z)$$

$$= \lim_{z\to bi}(z - bi) = \frac{1}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \frac{1}{(a^2-b^2)2bi}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_{c} f(z)dz = 2\pi i\,[-\frac{1}{2ai\,(a^2-b^2)} + \frac{1}{2bi\,(a^2-b^2)}]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[-\frac{1}{a} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a+b)(a-b)}\left(\frac{a-b}{ab}\right)$$

$$= \frac{\pi}{ab\,(a+b)}$$

$$(1) \implies \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\frac{\pi}{ab\,(a+b)}$$

$$= \frac{\pi}{2ab\,(a+b)}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

Now, $\int_{o}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)}$

$= \frac{1}{2}\int_c \frac{dz}{(z^2+a^2)^2}$ where C is the upper semi circle

$= \frac{1}{2}\int_c f(z)dz$ ... (1)

Where, $f(z) = \frac{1}{(z^2+a^2)^2}$

To find the poles, put $(z^2 + a^2)^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order 2 here $z = ai$ lies in the upper half of z – plane. Find the residue of the inside pole.

(i) When $z = ai$

$$[Res\ f(z)]_{z=ai} = \operatorname*{Lim}_{z\to ai} \frac{d}{dz}(z-ai)^2 f(z)$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2}\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai}\left[\frac{-2}{(z+ai)^3}\right]$$

$$= -\frac{2}{(2ai)^3} = -\frac{2}{-8a^3 i} = \frac{1}{4ia^3}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i \left(\frac{1}{4ia^3}\right)$$

$$= \frac{\pi}{2a^3}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx = \int_C \frac{z^2-z+2}{(z^4+10z^2+9)} dz, \text{ where C is the upper semi circle.}$$

$$= \int_c f(z)dz \qquad \dots(1)$$

Where, $f(z) = \frac{z^2-z+2}{(z^4+10z^2+9)}$

To find the poles, put $z^4 + 10z^2 + 9 = 0$

$\Rightarrow (z^2 + 1)(z^2 + 9) = 0$

$\Rightarrow z = \pm i, \pm 3i$ are poles of order one.

Here $z = i, 3i$ lies in the inside pole

Find the residue of the inside pole.

(i) When $z = i$

$$[Res\, f(z)]_{z=i} = \lim_{z\to i}(z-i)\, f(z)$$

$$= \lim_{z\to i}\left[(z-i)\frac{(z^2-z+2)}{(z+i)(z-i)(z^2+9)}\right]$$

$$= \lim_{z\to i}\left[\frac{(z^2-z+2)}{(z+i)(z^2+9)}\right]$$

$$= \frac{-1-i+2}{(2i)(8)} = \frac{1-i}{16i}$$

(ii) When $z = 3i$

$$[Res\, f(z)]_{z=3i} = \lim_{z\to 3i}\frac{d}{dz}(z-3i)\, f(z)$$

$$= \lim_{z\to 3i}\left[(z-3i)\frac{(z^2-z+2)}{(z^2+1)(z+3i)(z-3i)}\right]$$

$$= \lim_{z\to 3i}\frac{(z^2-z+2)}{(z^2+1)(z+3i)}$$

$$= \frac{-9-3i+2}{(-8)(6i)} = \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{3-3i+7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

$$(1) \Longrightarrow \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)}\,dx = \frac{5\pi}{12}$$

Example: Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$

Solution:

$$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$$

Replacement Put $x = z \Rightarrow dx = dz$

$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\int_C \frac{dz}{z^4+a^4}$ where C is the upper semi circle.

$$= \frac{1}{2}\int_c f(z)dz \quad \dots (1)$$

Where, $f(z) = \frac{1}{z^4+a^4}$

To find the poles, put $z^4 + a^4 = 0$

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = (-a^4)^{\frac{1}{4}}$$

$$\Rightarrow z = (-1)^{1/4}a$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}a$$

$$= [\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)]^{\frac{1}{4}}a$$

$$= \left[\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)\right]a$$

$$= ae^{+i\left(\frac{\pi+2k\pi}{4}\right)} ; k = 0, 1, 2, 3 \dots \dots$$

When $k = 0, z = ae^{\frac{i\pi}{4}}$

When $k = 1, z = ae^{\frac{3i\pi}{4}}$

When $k = 2, z = ae^{\frac{5i\pi}{4}}$

When $k = 3, z = ae^{\frac{7i\pi}{4}}$ are all poles of order one.

Here $z = ae^{\frac{i\pi}{4}}$ and , $z = ae^{\frac{3i\pi}{4}}$ lies in the upper half of the z plane.

Find the residue for the inside pole

(i)When $z = ae^{\frac{i\pi}{4}}$

$$[Res\, f(z)]_{z\to ae^{\frac{i\pi}{4}}} \left(z - ae^{\frac{i\pi}{4}}\right) f(z)$$

$$= \lim_{z\to ae^{\frac{i\pi}{4}}} \left[\left(z - ae^{\frac{i\pi}{4}}\right)\frac{1}{(z^4+a^4)}\right]$$

$$= \frac{0}{0}\ [Apply\ L'Hospital\ rule]$$

$$= \lim_{z\to ae^{\frac{i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{3i\pi}{4}}}$$

(ii) When $z = ae^{\frac{3i\pi}{4}}$

$$[Res\, f(z)]_{z=ae^{\frac{3i\pi}{4}}} = \lim_{z\to ae^{\frac{3i\pi}{4}}} \left(z - ae^{\frac{3i\pi}{4}}\right) f(z)$$

$$= \lim_{z\to ae^{\frac{3i\pi}{4}}} \left(z - ae^{\frac{3i\pi}{4}}\right) \frac{1}{z^4+a^4}$$

$$= \frac{0}{0}\ [Apply\ L'Hospital\ rule]$$

$$= \lim_{z\to ae^{\frac{3i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{9i\pi}{4}}}$$

∴ By Cauchy’s Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i \left(\frac{1}{4a^3 e^{\frac{3i\pi}{4}}} + \frac{1}{4a^3 e^{\frac{9i\pi}{4}}}\right)$$

$$= \frac{2\pi i}{4a^3}\left(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}}\right)$$

$$= \frac{\pi i}{2a^3}\left(e^{-\pi i} e^{\frac{i\pi}{4}} + e^{-i2\pi} e^{-\frac{i\pi}{4}}\right) \qquad [\because\ e^{-\pi i} = -1]$$

$$= \frac{\pi i}{2a^3}\left((-1)e^{\frac{i\pi}{4}} + (-1)e^{-\frac{i\pi}{4}}\right) \qquad \left[\because e^{-2\pi i} = -1\right]$$

$$= \frac{-\pi i}{a^3}\left(\frac{e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}}{2}\right) \qquad \left[\because \frac{e^{ix}-e^{-ix}}{2} = i\sin x\right]$$

$$= \frac{-\pi i}{a^3}\left(i \sin\frac{\pi}{4}\right)$$

$$= \frac{\pi}{a^3}\left(\frac{1}{\sqrt{2}}\right)$$

$$(1) \implies \int_o^\infty \frac{dx}{(x^4+a^4)} = \frac{1}{2}\left(\frac{\pi}{\sqrt{2}a^3}\right)$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \int_C \frac{dz}{(z^4+1)^3}$ where C is the upper semi circle.

$$= \int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{(z^4+1)^3}$

To find the poles, put $(z^2 + 1)^3 = 0$

$\Rightarrow z^2 + 1 = 0$

$\Rightarrow z = \pm i$ are poles of order 3.

Here $z = i$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = i$

$$[Res\, f(z)]_{z=i} = \frac{1}{2} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2}(z-i)^3 f(z)$$

$$= \frac{1}{2!} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2}\left[(z-i)^3 \frac{1}{(z+i)^3(z-i)^3}\right]$$

$$= \frac{1}{2!} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d}{dz}\left[\frac{-3}{(z+i)^4}\right]$$

$$= \frac{1}{2}\frac{12}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\, [sum\, of\, residues]$$

$$= 2\pi i \left(\frac{3}{16i}\right) = \frac{3\pi}{8}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \frac{3\pi}{8}$$

Type III

Integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(z)} \sin(nx)dx\ (or) \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos(nx)dx$$

To evaluate this integral, write $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} thus,

$$\int_C \frac{f(z)}{g(z)} e^{inz} dz = \int_{-\infty}^{\infty} \frac{d(z)}{g(x)} e^{inx} dx$$

Where C is the closed curve as in type II and finally equate imaginary part or real part accordingly to get the required integral.

Example: Evaluate $\int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx, a > 0, m > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$ and $\cos mn = R.P\ e^{imn}$

Now, $\int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{RP\ e^{imx}}{x^2+a^2} dx$

$= \frac{1}{2}\int_C \frac{RP\ e^{imz}}{z^2+a^2} dz$ where C is the upper semi circle.

$$= \frac{R.P}{2} \int_c f(z)dz \qquad \ldots (1)$$

Where $f(z) = \frac{e^{imz}}{z^2+a^2}$

To find the poles, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z \to ai} (z - ai)\, f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{e^{imz}}{(z+ai)(z-ai)}$$

$$= \lim_{z \to ai} \frac{e^{imz}}{(z+ai)}$$

$$= \frac{e^{-ma}}{2ai}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\, [sum\, of\, residues]$$

$$= 2\pi i \left(\frac{e^{-ma}}{2ai}\right)$$

$$= \frac{\pi\, e^{-ma}}{a}$$

$$(1) \Rightarrow \int_o^\infty \frac{\cos mx}{x^2+a^2} dx = \frac{R.P}{2}\left(\pi \frac{e^{-ma}}{a}\right) = \frac{\pi}{2a} e^{-ma}$$

Example: Evaluate $\int_o^\infty \frac{x \sin mx}{x^2+a^2} dx$ where $a > 0, m > 0$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz\ and\ \sin(mx) = IP\ e^{imx}$

Now, $\int_0^\infty \frac{x \sin mx}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{xIP\, e^{imx}}{x^2+a^2} dx$

$= \frac{I.P}{2} \int_c \frac{z\, e^{imz}}{z^2+a^2} dz$ where C is the upper semi circle.

$$= \frac{I.P}{2} \int_c f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{z\, e^{imz}}{z^2+a^2}$

To find the poles, put $f(z)$, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z \to ai} (z - ai)\, f(z)$$

$$= \lim_{z \to ai}(z - ai)\frac{ze^{imz}}{(z+ai)(z-ai)}$$

$$= \frac{(ai)e^{-ma}}{2ai} = \frac{e^{-ma}}{2}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\,of\,residues]$$

$$= 2\pi i\left(\frac{e^{-ma}}{2}\right) = \pi\,e^{-ma}$$

$$(1) \implies \int_o^\infty \frac{x\sin mx}{x^2+a^2}dx = \frac{I.P}{2}(\pi i\,e^{-ma}) = \frac{\pi}{2}e^{-ma}$$

Example: Evaluate $\int_{-\infty}^{\infty}\frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$

Solution:

Replacement $put\ n = z \Rightarrow dz = dz\cos x = R.P\ e^{ix}$

Now, $\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\frac{dx}{(x^2+b^2)} = \int_c \frac{RP\ e^{iz}\ dz}{(z^2+a^2)(z^2+b^2)}$

where C is the upper semi circle.

$$= \frac{R.P}{2}\int_c f(z)dz$$

Where, $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $f(z)$, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one here $z = ai, bi$ lies in the upper half of z – plane.

To find the residue for the inside pole

(i) when $z = ai$

$$[Res\,f(z)]_{z=ai} = \lim_{z \to ai}(z - ai)\,f(z)$$

$$= \lim_{z \to ai}(z - ai)\frac{e^{iz}}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \lim_{z \to ai}\frac{e^{iz}}{(z+ai)(z^2+b^2)}$$

$$= \frac{e^{-a}}{(2ai)(b^2-a^2)} = \frac{-e^{-a}}{(2ai)(a^2-b^2)}$$

(ii) when $z = bi$

$$[Res\,f(z)]_{z=bi} = \lim_{z \to bi}(z - bi)\,f(z)$$

$$= \lim_{z \to bi}(z - bi)\frac{e^{iz}}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \lim_{z \to bi}\frac{e^{iz}}{(z^2+a^2)(z+bi)}$$

$$= \frac{e^{-b}}{2bi(a^2-b^2)}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\left(\frac{e^{-a}}{(2ai)(a^2-b^2)} + \frac{e^{-b}}{(2bi)(a^2-b^2)}\right)$$

$$= \frac{2\pi i}{(2i)(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$(1) \Rightarrow \int_{-\infty}^{\infty}\frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)} R.P. \frac{\pi}{a^2-b^2}\left(\frac{ae^{-b}-be^{-a}}{ab}\right) = \frac{\pi}{ab\ (a^2-b^2)}(ae^{-b} - be^{-a})$$