4.1 AUTO CORRELATION FUNCTION

DEFINITION

Let {X(t)} be a random process. Then the auto correlation function of {Y(t)} is defined by $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$ (i.e) The auto correlation function of {X(t)} is the expected value of product of two samples of {X(t)}.

Properties of Auto Correlation Function

Property 1 : Auto Correlation function is even (or) Prove that $R_{XX}(-\tau) = R_{XX}(\tau)$

Proof:

$$R_{XX}(\tau) = E[X(t+\tau)X(t)]$$

$$R_{XX}(-\tau) = E[X(t-\tau)X(t)]$$

$$= E[X(t)X(t-\tau)]$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$\therefore R_{XX}(\tau)$ is an even function

Property 2 : Prove that $R_{XX}(0) = E[X^2(t)]$

Proof:

$$R_{XX}(\tau) = E[X(t+\tau)X(t)]$$

By taking $\tau = 0$, We get

$R_{XX}(0) = E[X(t)X(t)] = E[X^2(t)]$

$\therefore R_{XX}(0) = E[X^2(t)]$

Property 3 : Prove that $|R_{XX}(\tau)| \leq R_{XX}(0)$ (or) Prove that the maximum value of $R_{XX}(\tau)$ *is* $R_{XX}(0)$

Proof :

$$R_{XX}(\tau) = E[X(t+\tau)X(t)]$$

$$[R_{XX}(\tau)]^2 = \left[E[X(t+\tau)X(t)]\right]^2$$

$\leq E[X^2(t+\tau)]E[X^2(t)]$ by Schwartz inequality

$$= R_{XX}(0)R_{XX}(0)$$

$$\therefore |R_{XX}(\tau)|^2 \leq [R_{XX}(0)]^2$$

$$|R_{XX}(\tau)|^2 \leq [R_{XX}(0)]^2$$

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

Property 4 : If X(t) is a stationary process and it has no periodic component ,then $\mu_X = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$

Proof : Since {X(t)} is a stationary process, then as $\tau \to \infty, X(t) and\ X(t+\tau)$ are independent and $\mu_X = E[X(t)]$

$$R_{XX}(\tau) = E[X(t+\tau)X(t)]$$

$$\lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty} E[X(t+\tau)X(t)]$$

$= \lim_{\tau\to\infty} E[X(t+\tau)]E[X(t)]$

$= \lim_{\tau\to\infty} \mu_X\mu_X = {\mu_X}^2$

$$\Rightarrow \mu_X = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$$

$\therefore E[X(t)] = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$

PROBLEMS UNDER PROPERTIES OF AUTO CORRELATION FUNCTION

1. **Check whether the following are valid auto correlation functions:**
   (i) $\mathbf{f(\tau) = A\cos\omega\tau}$ **where A and ω are positive constants**
   (ii) $\mathbf{g(\tau) = A\sin\omega\tau}$ **where A and ω are positive constants**
   (iii) $\boldsymbol{h(\tau) = 1 - \frac{|\tau|}{T}}$ **where T is a constant.**

Solution :

(i) $f(\tau) = A\cos\omega\tau$

$f(-\tau) = A\cos(-\omega\tau)$

$= A\cos\omega\tau = f(\tau)$

$f(-\tau) = f(\tau)$

$\therefore$ $f(\tau)$ is the even function.

Hence $f(\tau)$ is the ACF of a process.

(ii) $g(\tau) = A\sin\omega\tau$

$g(-\tau) = A\sin(-\omega\tau)$

$= -A\sin\omega\tau = -g(\tau)$

$g(-\tau) = -g(\tau)$

$\therefore$ $g(\tau)$ is an odd function.

Hence g(τ) is not a ACF of a process.

(iii) $h(\tau) = 1 - \frac{|\tau|}{T}$

$h(-\tau) = 1 - \frac{|-\tau|}{T}$

$= 1 - \frac{|\tau|}{T} = h(\tau)$

$h(-\tau) = h(\tau)$

∴ h(τ) is the even function.

Hence h(τ) is the ACF of a process.

2. **Let {X(t)} be a WSS process with $R(\tau) = 25 + \frac{1}{1+6\tau^2}$. Find the mean, mean square value and variance of {X(t)}**

Solution:

The mean value of {X(T)} is given by

$$E[X(t)] = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$$

$$= \sqrt{\lim_{\tau\to\infty} (25 + \frac{1}{1+6\tau^2})}$$

$$= \sqrt{25+0} = \sqrt{25}$$

$$\therefore Mean\ E[X(t)] = 5$$

The mean square value of {X(t)} is given by

$$E[X^2(t)] = R_{XX}(0) = 25 + \frac{1}{1+6(0)} = 25 + 1 = 26$$

The Variance of {X(t)} is given by

$$Var[X(t)] = E[X^2(t)] - E[X(t)]^2$$

$= 26 – 5^2 = 26 – 25 = 1$

Variance $\sigma^2 = 4$

3. **A Stationary process has an auto correlation function is given by $R(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$. Find mean ,mean square value and variance of { X(t)}**

Solution :

The mean value of {X(T)} is given by

$$E[X(t)] = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$$

$$= \sqrt{\lim_{\tau\to\infty} \left(\frac{25\tau^2+36}{6.25\tau^2+4}\right)}$$

$$\sqrt{\lim_{\tau\to\infty} \frac{\tau^2(25+\frac{36}{\tau^2})}{\tau^2(6.25+\frac{4}{\tau^2})}} = \sqrt{\frac{25}{6.25}} = \sqrt{4} = 2$$

$$\therefore Mean\ E[X(t)] = 2$$

The mean square value of {X(t)} is given by

$E[X^2(t)] = R_{XX}(0)$

$= \frac{36}{4} = 9$

The variance of {X(t)} is given by

$Var\{X(t)\} = E[X^2(t)] – E[X(t)]^2$

$= 9 – 2^2 = 9 - 4 = 5$

Variance $\sigma^2 = 5$

4. **Let {X(t)} be a stationary process with $R(\tau) = 2 + 6e^{-3|\tau|}$. Find the mean and variance of {X(t)}**

Solution :

The mean value of {X(T)} is given by

E[X(t)] = $\sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$

= $\sqrt{\lim_{\tau\to\infty} (2+6e^{-3|\tau|})}$

$\sqrt{2+6e^{-|\infty|}}$ = $\sqrt{2+0}$ = $\sqrt{2}$

$\therefore Mean\ E[X(t)] = \sqrt{2}$

The mean square value of {X(t)} is given by

E[X^2 (t)] = $R_{XX}(0)$

$= 2+6e^{-|0|} = 2+6 = 8$

The variance of {X(t)} is given by

Var{X(t)} = E[X^2(t)] – $E[X(t)]^2$

= 8 – $\sqrt{2}^{\,2}$ = 8 - 2 = 6

Variance σ^2 = 6

5. **If {X(t)} is a WSS process with auto correlation function $R_{XX}(\tau) = 4e^{-2|\tau|} + \frac{9}{4\tau^2+3}$. Find the mean and variance of Y = X(4) – X(2)**

Solution :

Given $R(\tau) = 4e^{-2|\tau|} + \frac{9}{4\tau^2+3}$ …….(1)

First we have to find the mean and variance of {X(t)}

The mean value of {X(t)} is given by

E[X(t)] = $\sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$

$= \sqrt{\lim_{\tau\to\infty} (4e^{-2|\tau|} + \frac{9}{4\tau^2+3})}$

$= \sqrt{0+0} = \sqrt{0}$

$\therefore Mean\ E[X(t)] = 0 \dots\dots(2)$

The mean square value of {X(t)} is given by

$E[X^2(t)] = R_{XX}(0)$

$= 4 + 3 = 7$ from (1)

The variance of {X(t)} is given by

$Var\{X(t)\} = E[X^2(t)] - E[X(t)]^2$

$= 7 - 0^2 = 7$

Variance $\sigma^2 = 7$……..(3)

We have to find mean and variance of Y = X(4) – X(2)

(i) Mean of Y = E[Y] = E[X(4) – X(2)]

= E[X(4)] – E [X(2)]

= 0 - 0 = 0

(ii) Var(Y) = Var [X(4) – X(2)]

= Var[X(4)] + Var[x(2)] – 2Cov[X(4) . X(2)]

= 7 + 7 – 2 Cov (4,2)…………….(4)

We know that Cov (4,2) = R_{XX} (4,2) – E[X(4)] E[X(2)]

= R_{XX} (4 - 2) – 0 = R_{XX} (2) From (2)

$= 4e^{-2(2)} + \frac{9}{16+3}$

$= 4e^{-2(2)} + \frac{9}{19}$ from (1)

= 0.546

(4) ⇒ Var (Y) = 14 – 2(0.546)

= 14 – 1.092

∴ Var(Y) = 12.908

6. **Suppose that { X(t)} is a WSS process with ACF R(t_1, t_2) = 9 + 4 $e^{-0.2|t_1-t_2|}$.Detemine the mean,variance and the covariance of the RV'S Z = X(5) and W = X(8).**

Solution :

Given $R(\tau) = 9 + 4\,e^{-0.2|\tau|}$

We have to find the mean and variance {X(t)}

(i) The mean value of {X(t)} is given by

E[X(t)] = $\sqrt{\lim_{\tau\to\infty} R(\tau)}$

= $\sqrt{\lim_{\tau\to\infty}[9 \; + \; 4\text{ e} - 0.2|\tau|]}$

= $\sqrt{9 + 4e^{-\infty}}$

= $\sqrt{9 + 0}$

$\therefore Mean\ E[X(t)] = 3 \ldots \ldots (1)$

(ii) The mean square value of {X(t)} is given by

E [X^2(t)] = R(0)

= 9 + 4 e^0 = 13.

The variance of {X(t)} is given by

Var[X(t)] = $\sigma^2_{X(t)}$ = E[X^2(t)] – $E[X(t)]^2$

= 13 – 3^2 = 4……….(2)

(iii) Given Z = X(5) ; W = X(8)

Mean of Z = E(Z) = E[X(5)] = 3 from(1)

Mean of W = E(W) = E[X(8)] = 3 from(1)

Var(Z) = Var[X(5)] = 4 from (2)

Var (W) = Var[X(8)] = 4 from (2)

Cov(Z,W) = E[ZW] – E[Z]E[W]

= E[X(5) X(8)] – 3 X 3

= R(5,8) - 9 = 9 + 4 $e^{-0.2|-3|} - 9$

= 4 $e^{-0.6}$

7. **If [X(t)] is a wide sense stationary process with auto correlation function $R(\tau) = Ae^{-\alpha|\tau|}$, Determine the second order moment of the RV X(8) – X(5)**

Sol:

Given $R(\tau) = Ae^{-\alpha|\tau|}$ ______(1)

$E[X^2(t)] = R(0) = Ae^0 = A$ ….. ….. ….. (2)

The second order moment of X(8) - X(5) is given by

$=E[(X(8) - X(5))^2]$

$=E[X^2(8)] + X^2(5) - 2X(8)X(5)]$

$=E[X^2(8)] + E[X^2(5)] - 2E[X(8)X(5)]$

$= A + A - 2R_{XX}(8,5)$ from(2)

$= 2A - 2R_{XX}(8,5)$

$= 2A - 2R_{XX}(3)$

$= 2A - 2Ae^{-\alpha|3|}$ from(1)

$=2A(1 - e^{-3\alpha})$

8. **If {X(t)} is a WSS process, then prove that $E[X(t+\tau) - X(t)]^2 = 2[R_{XX}(0) - R_{XX}(\tau)]$**

Proof :

$$E[X(t+\tau) - X(t)]^2 = E[X^2(t+\tau) + X^2(t) - 2\,X(\,t+\tau)X(t)]$$
$$= E[X^2(t+\tau)] - E[X^2(t)] - 2E[\,X(\,t+\tau)X(t)]$$
$$= R_{XX}(0) + R_{XX}(0) - 2R_{XX}(\tau)$$
$$= 2R_{XX}(0) - 2R_{XX}(\tau)$$
$$E[X(t+\tau) - X(t)]^2 = 2[R_{XX}(0) - R_{XX}(\tau)]$$

9. **Suppose that {X(t}is a WSS process with mean that $\mu_X \neq 0$ and that Y(t) is defined by Y(t)= X(t+τ) - X(t), where τ > 0 is a constant. Show that the mean of [Y(t)] is zero for all value of t and the variance of [Y(t)] is given by σ_Y^2= 2[R_{XX}(0) - $R_{XX}(\tau)$]. Is [Y(t)] a WSS process?**

Sol:

Given: Y(t) = X(t +τ) - X(t)

Given {X(t)} is a WSS process.

∴(1) E[X(t)] = constant = μ_X and

∴(2) $R_{XX}(t_1,t_2)$ is a function of τ.

(**i**) Mean of Y(t) = E[Y(t)]

=E[X(t +τ) - X(t)]

=E[X(t + τ)] - E[X(t)]

=μ_X- μ_X= 0

(ii) $\sigma^2_{Y(t)}$ = Var[Y(t)]

= Var[X(t +τ) -X(t)]

= Var[X(t + τ)] + Var[X(t)] - 2 Cov[X(t + τ) X(t)]

= σ^2_X(t) +σ^2_X(t) - 2{E[(X(t +τ)X(t)] -E[X(t+τ)]E[X(t)]}

= 2σ^2_X(t) - 2[$R_{XX}(\tau)$- $\mu_X\mu_X$]

= 2[σ^2_X(t) - $R_{XX}(\tau)$+μ^2_X

=2[E[X^2 (t)] - $[E[X(t)]]^{2]}$ - $R_{XX}(\tau)$ + μ^2_X]

=2[E(X^2 (t) - μ^2_X - $R_{XX}(\tau)$+μ^2_X)]

=2[$R_{XX}(0)$- $R_{XX}(\tau)$]

(iii) $R_{YY}(t_1,t_2)$ = E[Y(t_1)Y(t_2)]

=E[[X(t_1+ τ) - X(t_1)] [X(t_1 + τ) - X(t_1)]]

=E[X(t_1+ τ)X(t_2+ τ) - X(t_1+ τ)X(t_2) - X(t_1)X(t_1+ τ) + X(t_1) X(t_2)]

=E[X(t_1+ τ)X(t_2+ τ)] - E[X(t_1+ τ)X(t_2)] - E[X(t_1)X(t_1+ τ)] + E[X(t_1) X(t_2)]

=$R_{XX}(t_1+ \tau- t_2+ \tau)$ - $R_{XX}(t_1+ \tau- t_2)$ -$R_{XX}(t_1- t_2- \tau)$ + $R_{XX}(t_1- t_2)$

=$R_{XX}(\tau)$ - $R_{XX}(\tau+ \tau)$ - $R_{XX}(0)$ +$R_{XX}(\tau)$

$R_{YY}(t_1,t_2)$ = 2$R_{XX}(\tau)$ - $R_{XX}(2\tau)$ - $R_{XX}(0)$

Which is a function of τ.

Also we have E[Y(t)] = 0

∴ [Y(t)] a WSS process

MEAN ERGODIC THEOREM :

Let {X(t)} be a random process with constant mean μ_X. Then {X(t)} is mean ergodic if $\lim_{T\to\infty} Var(\overline{X_T}) = 0$.

1. **The auto correlation function for a stationary process {X(t)} is given by $R_{XX}(\tau)$ = 9 +2$e^{-|\tau|}$. Find the Mean of the random variable Y = $\int_0^2 X$(t)dt and the variance of {X(t)}.**

Sol:

The mean of {X(t)} is given by

$$E[X(t)] = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$$

$$= \sqrt{\lim_{\tau\to\infty}[9 + 2e^{-|\tau|}]} = \sqrt{9} = 3$$

The mean Square value of {X(t)} is given by

$$E[X^2(t)] = R_{XX}(0) = 9+2 = 11$$

The variance if {X(t)} is given by

$$Var[X(t)] = \sigma_X^2 = E[X^2(t)] - [E[X(t)]]^2 = 11 - 9 = 2$$

Next, we find the mean of Y as

$E[Y] = E[\int_0^2 X(t)\, dt]$

$= \int_0^2 E\, [X(t)]\, dt$

$= \int_0^2 3\, dt$

$= 3[t]_0^2$

$= 6$

2. **The auto correction function of for a stationary process [X(t)] is given by $R_{XX}(\tau) = 1 + e^{-2|\tau|}$. Find the mean and variance of S = $\int_0^1 X(t)dt$.**

Sol.

X(t) is defined in (0,1). ∴T=1

Given $R(\tau) = 1+e^{-2|\tau|}$

The mean of [x(t)] is given by

$E[X(t)] = \sqrt{\lim_{\tau\to\infty} R(\tau)} = \sqrt{\lim_{\tau\to\infty}[1 + e^{-2|\tau|}]} = 1$

Given S = $\int_0^1 X(t)dt$.

We have to Find the mean and variance of S

(i) The mean of S is given by

$E[S] = \int_0^1 E[X(t)]dt = \int_0^1 (1)dt = [t]_0^1 = 1\text{-}0$

$\therefore$E[S] = 1

(ii) To compute variance of S.

$\overline{X_T} = \frac{1}{T}\int_0^T X(t)\,dt = \int_0^1 X(t)dt = S$ [$\because$ T = 1]

$Var(\overline{X_T}) = \frac{1}{T}\int_{-T}^{T}(1-\frac{|\tau|}{T})\, C_{XX}(\tau)d\tau$ Here T = 1

$= \int_{-1}^{1}.(1-|\tau|)\, C_{XX}(\tau)\, d\tau$ ….. …… …… (1)

$C_{XX}(\tau) = R_{XX}(\tau)$ - E[X(t_1)] E[X(t_2)]

$= 1 + e^{-2|\tau|} - 1 \times 1$ [$\because$E[X(t)] = 1]

$= e^{-2|\tau|}$

(1) $\Rightarrow$ $Var(\overline{X_T}) = \int_{-1}^{1} (1-|\tau|)e^{-2|\tau|}d\tau = 2\int_0^1.(1-|\tau|)e^{-2|\tau|}d\tau$

$= 2\int_0^1 (1-\tau)e^{-2\tau}d\tau = 2[(1-\tau)[\frac{e^{-2\tau}}{2}] - (-1)\,(\frac{e^{-2\tau}}{(-2)^2})]_0^1$

$= 2[\,0 + \frac{e^{-2}}{4} + \frac{1}{2} - \frac{1}{4}] = 2[\frac{e^{-2}}{4} + \frac{1}{4}] = \frac{e^{-2}}{2} + \frac{1}{2}$

$Var(\overline{X_T}) = \frac{1}{2}(1 + e^{-2}) = 0.5677$

Since S = $\overline{X_T}$, Var(S) = 0.5677

3. If S = $\int_0^{10} X(t)\,dt$, Find the mean and variance of S if E[X(t)] = 8 and $R_{XX}(\tau) = 64 + 10e^{-2|\tau|}$

Sol:

X(t) is defined in (0,10) $\quad$ *since* T = 10

Given: S = $\int_0^{10} X(t)\,dt$; E[X(t)] = 8 ; $R_{XX}(\tau) = 64 + 10e^{-2|\tau|}$

(i) Mean of S, E(S) = $\int_0^{10} E\,[X(t)]\,dt = \int_0^{10} 8\,\,dt = 8[t]_0^{10} = 80$

(ii) To compute Variance of $\overline{X_T}$

$\overline{X_T}\ \frac{1}{10}\int_0^{10} X(t)\,dt = \frac{S}{10}$

$Var(\overline{X_T}) = \frac{1}{T}\int_{-T}^{T}(1-\frac{|\tau|}{T})\, C_{XX}(\tau)d\tau$….. ….. …… (1)

$C_{XX}(\tau) = R_{XX}(\tau) - E[X(t_1)]\,E[X(t_2)]$

$=(64 + 10e^{-2|\tau|}) - (8 \times 8)$

$=10e^{-2|\tau|}$

(1) ⇒ $Var(\overline{X_T}) = \frac{1}{10}\int_{-10}^{10}(1-\frac{|\tau|}{10})\, 10e^{-2|\tau|}d\tau = \int_{-10}^{10}(1-\frac{|\tau|}{10})\, e^{-2|\tau|}d\tau$

$=2\int_0^{10}(1-\frac{|\tau|}{10})\, e^{-2|\tau|}d\tau = 2\int_0^{10}(1-\frac{\tau}{10})\, e^{-2\tau}d\tau$

$=2[(1-\frac{\tau}{10})(\frac{e^{-2\tau}}{-2}) - (\frac{-1}{10})\frac{e^{-2\tau}}{(-2)^2}]_0^{10}$

$= 2[0 + \frac{1}{40}e^{-20} + \frac{1}{2} - \frac{1}{40}] = 2[\frac{e^{-20} + 20 - 1}{40}]$

$Var(\overline{X_T}) = \frac{1}{20}[e^{-20} + 19] = 0.95$ …. …. ….. (1)

We have $\overline{X_T} = \frac{S}{10} \Rightarrow 10\overline{X_T}$

$Var(S) = Var(10\overline{X_T}) = 10^2 Var(\overline{X_T})$

$= 100 \times 0.95$ From (1)

$Var(S) = 95$

4. **The random binary transmission process {X(t)} is a wide sense process with zero mean and auto correlation function R(τ) = 1-$\frac{|\tau|}{T}$ Where T is a constant. Find the mean and variance of the time average of {x(t)} over (0,T). Is {X(t)} mean-ergodic?**

Sol: Given mean of {X(t)} is zero.

$\therefore E[X(t)] = 0$

The time average of {X(t)} is over (0, T) given by

$\overline{X_T} = \frac{1}{T}\int_0^T X(t)\,dt$

The mean value of time average of {X(t)} is given by

$E(\overline{X_T}) = \frac{1}{T}E\left[\frac{1}{T}\int_0^T [X(t)]\,dt\right]$

$= \frac{1}{T}\int_0^T E[X(t)]\,dt$ $\because E[X(t)] = 0$

$= 0$

The variance of time average of [X(t)] is given by

$Var(\overline{X_T}) = \frac{1}{T}\int_{-T}^{T}(1 - \frac{|\tau|}{T})\, C_{XX} d\tau$…. …. …. (1)

Given $R_{XX}(\tau)$ = 1- $\frac{|\tau|}{T}$;$|\tau| \leq$ T

$C_{XX}(\tau)$ = $R_{XX}(\tau)$ - E[X(t)]E[X(t + τ)]

=$R_{XX}(\tau)$ - 0

= 1- $\frac{|\tau|}{T}$;$|\tau| \leq$ T

(1) ⇒ Var($\overline{X_T}$)= $\frac{1}{T} \int_{-T}^{T}(1 - \frac{|\tau|}{T})^2 d\tau$

$= \frac{2}{T} \int_{0}^{T}(1 - \frac{|\tau|}{T})^2 d\tau$

$= \frac{2}{T} \int_{0}^{T}(1 - \frac{\tau}{T})^2 d\tau$

$= \frac{2}{T} [\frac{(1-\frac{|\tau|}{T})^3}{3(-\frac{1}{T})}]_0^T$

= - $\frac{2}{3}$[0-1] = $\frac{2}{3}$

$$\lim_{T\to\infty} \text{Var}(\overline{X_T}) = \frac{2}{3} \neq 0$$

$\therefore$ $[X(t)]$ *is not mean ergodic by mean ergodic theorem.*