## 1.5 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY STRAIN ENERGY METHOD

## (PLANE TRUSSES)

### 1.5.1 STATIC INDETERMINACY OF STRUCTURES

If the number of independent static equilibrium equations (refer to Section 1.2) is not sufficient for solving for all the external and internal forces (support reactions and member forces, respectively) in a system, then the system is said to be statically indeterminate.

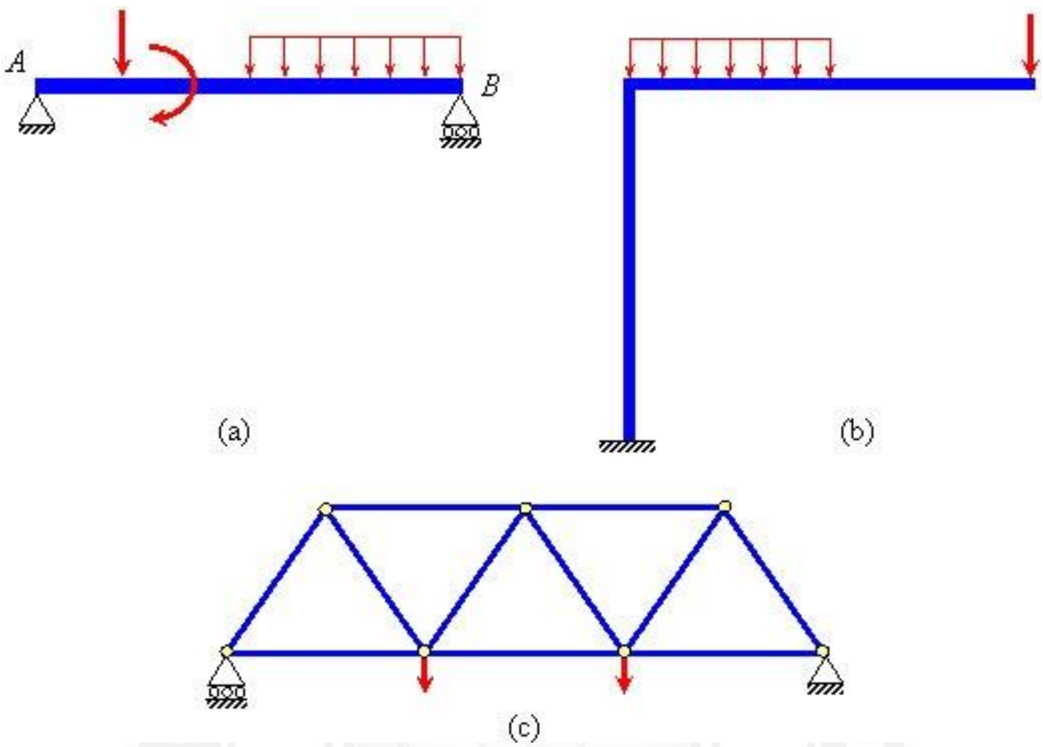
A statically determinate system, as against an indeterminate one, is that for which one can obtain all the support reactions and internal member forces using only the static equilibrium equations.

For example, for the system in Figure 1.10, idealized as one-dimensional, the number of independent static equilibrium equations is just 1 ( $R_A \& R_B$ ), while the total number of unknown support reactions are 2 ( $\sum F_x = 0$ ), that is more than the number of equilibrium equations available.

Therefore, the system is considered statically indeterminate. The following figures illustrate some example of statically determinate (Figures 1.11a-c) and indeterminate structures (Figures 1.12a-c).

In Section 1.2, the equilibrium equations are described as the necessary and sufficient conditions to maintain the equilibrium of a body. However, these equations are not always able to provide all the information needed to obtain the unknown support reactions and internal forces.

The number of external supports and internal members in a system may be more than the number that is required to maintain its equilibrium configuration. Such systems are known as indeterminate systems and one has to use compatibility conditions and constitutive relations in addition to equations of equilibrium to solve for




the unknown forces in that system.

Figure 1.11 Statically determinate structures

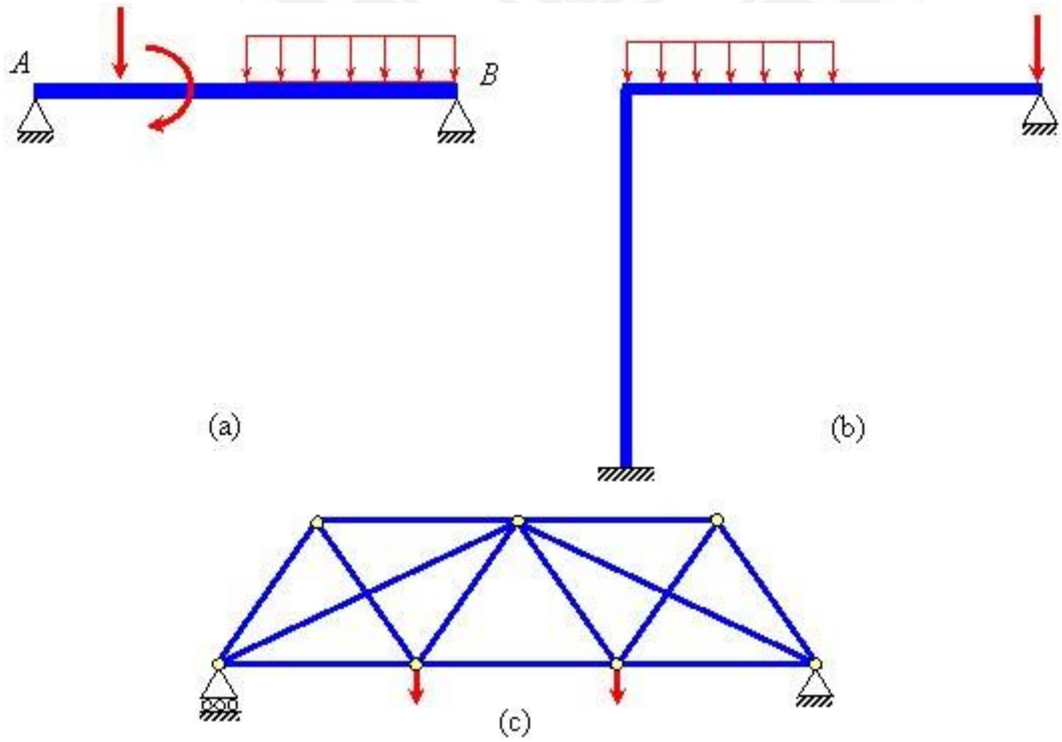



Figure 1.12 Statically indeterminate structures

For an indeterminate system, some support(s) or internal member(s) can be removed without disturbing its equilibrium. These additional supports and members are known as redundants . A determinate system has the exact number of supports and internal members that it needs to maintain the equilibrium and no redundants. If a system has less than required number of supports and internal members to maintain equilibrium, then it is considered unstable.

For example, the two-dimensional propped cantilever system in (Figure 1.13a) is an indeterminate system because it possesses one support more than that are necessary to maintain its equilibrium. If we remove the roller support at end B (Figure 1.13b), it still maintains equilibrium. One should note that here it has the same number of unknown support reactions as the number of independent static equilibrium equations. The unknown

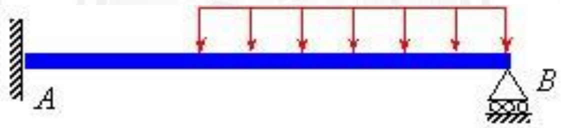



Figure 1.13a Propped cantilever $AB$

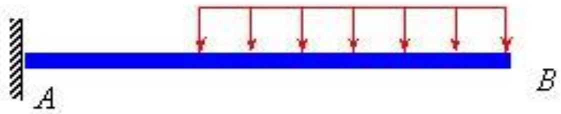



Figure 1.13b Cantilever $AB$ with no roller support at $B$

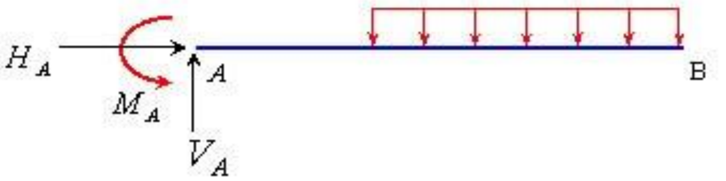



Figure 1.13c Free body diagram of cantilever $AB$ (with no roller support at $B$)

The reactions are $H_A$ ,$V_A$, & $M_A$ (Fig.1.13.c) and the equilibrium equations are;

$$\sum F_x = 0 \quad \text{-----} \quad (1.18)$$

$$\sum F_y = 0 \qquad ----- \quad (1.19)$$

$$\sum M_z (\text{about any point}) = 0 \qquad ---- \quad (1.20)$$

An indeterminate system is often described with the number of redundant, it posses and this number is known as its degree of static indeterminacy.
Thus,mathematically:

Degree of Static Indeterminacy = Total number of unknowns (External & Internal)

- number of independent equations of equilibrium

---- (1.21)

It is very important to know exactly the number of unknown forces and the number of independent equilibrium equations. Let us investigate the determinacy/indeterminacy of a few two-dimensional pin-jointed truss systems.

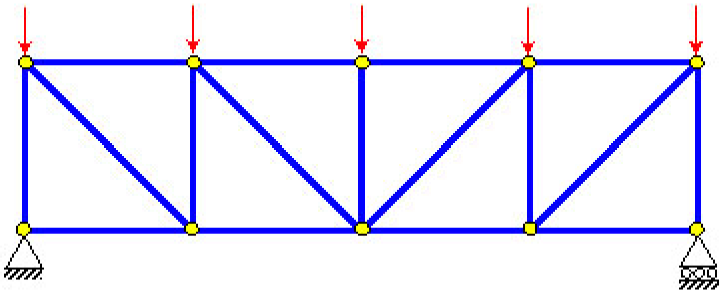
Let *m* be the number of members in the truss system and *n* be the number of pin (hinge) joints connecting these members. Therefore, there will be *m* number of unknown internal forces (each is a two-force member) and 2 *n* numbers of independent joint equilibrium equations ( $\sum F_x = 0$ and $\sum F_y = 0$ for each joint, based on its free body diagram). If the support reactions involve *r* unknowns, then:

Total number of unknown forces = $m + r$

Total number of independent equilibrium equations = 2 $n$

So, degree of static indeterminacy = ( $m + r$ ) - 2 $n$

For the trusses in Figures 1.14a, b & c, we have:



Figure; 1.14a  Determinate truss

**1.14a:** *m* = 17, *n* = 10, and *r* = 3. So, degree of static indeterminacy = 0, that means it is a statically determinate system.

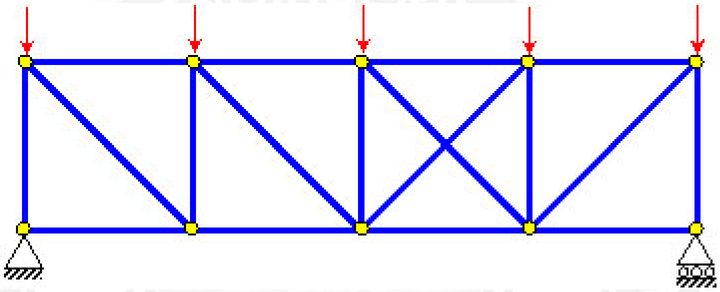


Figure 1.14b (Internally) indeterminate truss

**1.14b:** *m* = 18, *n* = 10, and *r* = 3. So, degree of static indeterminacy = 1.

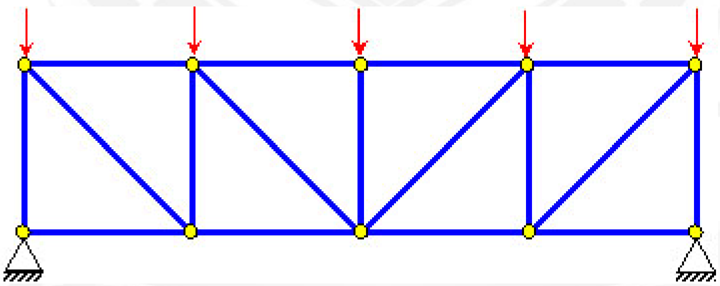


Figure 1.14c (Externally) indeterminate truss

**1.14c:** *m* = 17, *n* = 10, and *r* = 4. So, degree of static indeterminacy = 1.

It should be noted that in case of 1.14b, we have one member more than what is needed for a determinate system (i.e., 1.14a), where as 1.14c has one unknown reaction component more than what is needed for a determinate system. Sometimes, these two different types of redundancy are treated differently; as internal indeterminacy and external indeterminacy . Note that a structure can be indeterminate either externally or internally or both externally and internally.

We can group external and internal forces (and equations) separately, which will help us understand easily the cases of external and internal indeterminacy. There are $r$ numbers of external unknown forces, which are the support reactions components. We can treat 3 system equilibrium equations as external equations. This will lead us to:

Degree of external static indeterminacy = $r$ - 3.

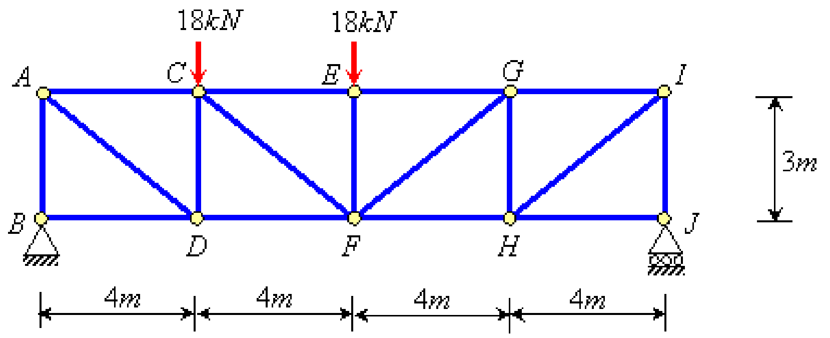
The number of internal unknown forces is $m$ and we are left with (2 $n$ -3) equilibrium equations. The 3 system equilibrium equations used earlier were not independent of joint equilibrium equations, so we are left with (2 $n$ - 3) equations instead of 2 $n$ numbers of equations.So:
Degree of internal static indeterminacy = $m$ - (2 $n$ - 3).

Please note that the above equations are valid only for two-dimensional pin-jointed truss systems. For example, for three-dimensional ( "space" ) pin-jointed truss systems, the degree of static indeterminacy is given by ( $m$ + $r$ - 3 $n$). Similarly, the expression will be different for systems with rigid (fixed) joints, frame members, etc.

**Examble problems on Trusses;**

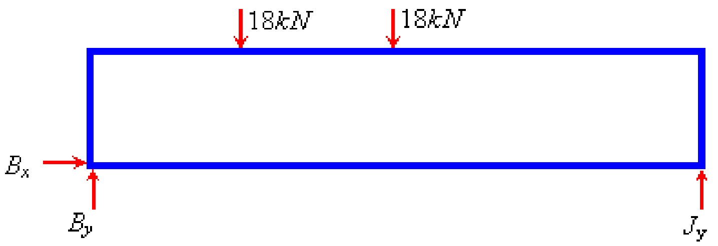
**Problemn no:01**

Find the forces in AB , AD and AC in the following Figure E2.1.



**Solution:**

FBD of the whole system

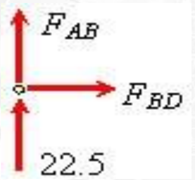



$$\sum F_x = B_x = 0$$

$$\sum M(about\,B) = J_y(16) - 18(8) - 18(4) = 0 \Rightarrow J_y = 13.5\,kN$$

$$\sum F_y = B_y + J_y - 18 - 18 = 0 \Rightarrow B_y = 22.5\,kN$$

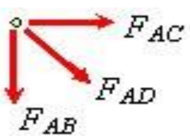
FBD of joint B;




$$\sum F_x = F_{BD} = 0$$

$$\sum F_y = 22.5 + F_{AB} = 0 \Rightarrow F_{AB} = -22.5\,kN$$

FDB of the joint A;




$$\sum F_y = -F_{AB} - \frac{3}{5}F_{AD} = 0 \qquad \Rightarrow F_{AD} = 37.5\,kN$$

$$\sum F_x = -F_{AC} + \frac{4}{5}F_{AD} = 0 \qquad \Rightarrow F_{AC} = -30\,kN$$

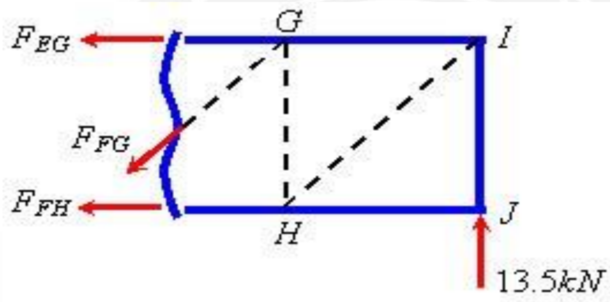
Force in *AB* = 22.5 *kN* (Compression).

Force in *AD* = 37.5 *kN* (Tension)

Force in *AC* = 30.0 *kN* (Compression).

**Problemn no:02**

Find the forces in *EG* , *FG* and *FH* in the following Figure E2.1.




$$\sum F_y = 13.5 - \frac{3}{5} F_{FG} = 0 \Rightarrow F_{FG} = 22.5\ kN$$

$$\sum M(about\ G) = 13.5(4) - F_{FH}(3) = 0 \quad \Rightarrow F_{FH} = 18\ kN$$

$$\sum F_x = -F_{EG} - \frac{4}{5} F_{FG} - F_{FH} = 0 \quad \Rightarrow F_{EG} = -36\ kN$$

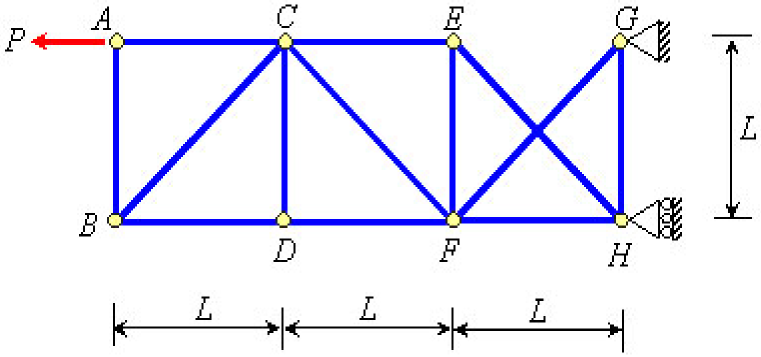
Force in EG = 36.0 *kN* (Compression).

Force in FG = 22.5 *kN* (Tension).

Force in FH = 18.0 *kN* (Compression).

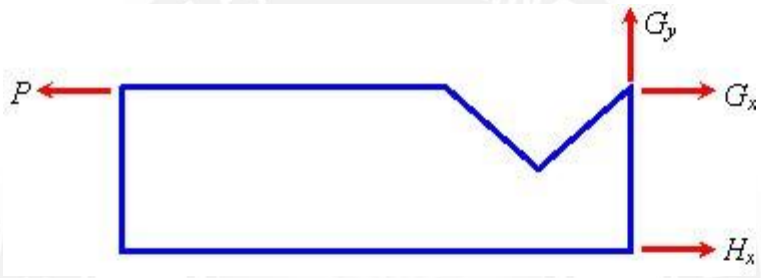
**Problemn no:03**

Find the forces in all members in the Figure E2.2.



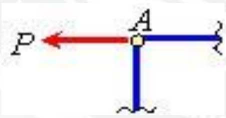

**Solution:**

From equilibrium of the whole body;



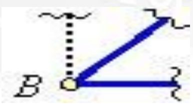

$$G_x = P,\ G_y = 0,\ H_x = 0$$

Looking at joint A :



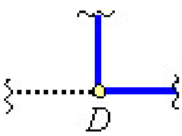

AB is a zero-force member and FAC = P

Looking at joint B :



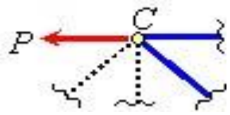

Both BC and BD are zero-force member.

Looking at joint D :



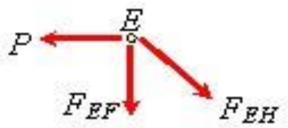

Both CD and DF are zero-force member.

Looking at joint C :



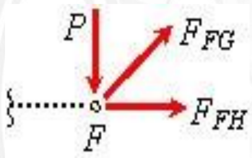

*CF* is a zero-force member and FCE= P

Equilibrium of joint B :



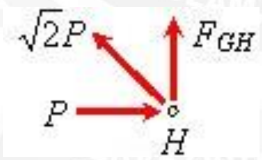

$F_{EF} = -P$ and $F_{EH} = \sqrt{2}P$

Equilibrium of joint F :




$F_{EF} = -P$ and $F_{EH} = \sqrt{2}P$

Equilibrium of joint H :

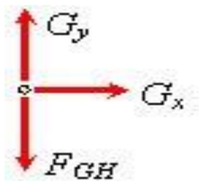


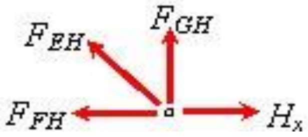

$F_{GH} = -P$

Sign convention: Tension +ve, compression –ve.

**Note:**

That we have not obtained the support reactions before finding the member forces. It was not necessary for this specific problem. Find out these reactions at supports *G* and *H* and check if joint equilibrium is satisfied at these two joints with the member forces that we have found already.


$G_y$
$G_x$
$F_{GH}$


$F_{GH}$
$F_{EH}$
$F_{FH}$
$H_x$