PROPERTIES OF LAPLACE TRANSFORM

Property: 1 Linear property

$\boldsymbol{L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]}$ **, where a and b are constants.**

Proof:

$$L[af(t) \pm bg(t)] = \int_0^\infty [af(t) \pm bg(t)]\, e^{-st} dt$$

$$= a\int_0^\infty f(t) e^{-st} dt \pm b\int_0^\infty g(t)\, e^{-st} dt$$

$$\boxed{L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]}$$

Property: 2 Change of scale property.

If $\boldsymbol{L[f(t)] = F(s)}$, **then** $\boldsymbol{L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}\ ; \boldsymbol{a} > 0$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st}\, f(t)\, dt = F(s) \cdots\cdots (1)$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^\infty e^{-st}\, f(at)\, dt \cdots\cdots (2)$$

Put at= x $ie., t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$(2) \Rightarrow L[f(at)] = \int_0^\infty e^{\frac{-sx}{a}}\, f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{\frac{-sx}{a}}\, f(x) dx$$

Replace x by t, $L[f(at)] = \frac{1}{a}\int_0^\infty e^{\frac{-st}{a}}\, f(t) dt$

$$\boxed{\boldsymbol{L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}\ ; \boldsymbol{a} > 0}$$

Property: 3 First shifting property.

If $\boldsymbol{L[f(t)] = F(s)}$**, then i)** $\boldsymbol{L[e^{-at}f(t)] = F(s+a)}$

ii) $\boldsymbol{L[e^{at}f(t)] = F(s-a)}$

Proof:

(i) $L[e^{-at}f(t)] = F(s+a)$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st}\, f(t)\, dt = F(s) \cdots (1)$$

By the definition of Laplace transform, we have

$$L[e^{-at}f(at)] = \int_0^\infty e^{-st}\, e^{-at} f(t)\, dt$$

$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$

$= F(s+a)$ by (1)

(ii) $L[e^{at} f(at)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$

$= F(s-a)$ by (1)

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$u = e^{-st}$

$\therefore du = -se^{-st} dt$

$dv = f'(t) dt$

$\therefore v = \int f'(t) dt$

$= f(t)$

$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty u dv$

$= [uv]_0^\infty - \int u dv$

$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)\, (-s) e^{-st}\, dt$

$= 0 - f(0) + sL[f(t)]$

$= sL[f(t)] - f(0)$

$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)}$

Property: 5 Laplace transform of derivative of order n

$\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \cdots\cdots (1)$

$L[f^n(t)] = L[[f'(t)]']$

$= sL[f'(t)] - f'(0)$

$= s[sL[f(t)] - f(0)] - f'(0)$

$= s^2 L[f(t)] - sf(0) - f'(0)$

Similarly, $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

In general, $\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$

Laplace transform of integrals

Theorem: 1 If$L[f(t)] = F(s)$, then $L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t) dt$

$\therefore g'(t) = f(t)$

And $g(0) = \int_0^0 f(t) dt = 0$

Now$L[g'(t)] = L[f(t)]$

$$sL[g(t)] - g(0) = L[f(t)]$$

$$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$$

$$L[g(t)] = \frac{L[f(t)]}{s}$$

$$\therefore \boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}}$$

Theorem: 2 If $\boldsymbol{L[f(t)] = F(s)}$, then $\boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\, dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty (-t)e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$-\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$-L[tf(t)] = \frac{d}{ds}F(s)$$

$$\therefore \boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$$

Note: In general $\boldsymbol{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)}$

Example: If $\boldsymbol{L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}}$ then find $\boldsymbol{L[f(2t)]}$.

Solution:

Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$$L[f(2t)] = \frac{1}{2}F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2}+1}{\left(2\,\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

$$= \frac{1}{2}\frac{\left[\frac{s^2}{4} - \frac{s}{2}+4\right]}{(s+1)^2\left(\frac{s-2}{2}\right)}$$

$$= \frac{s^2-2s+1}{4(s+1)^2(s-2)}$$