DIAGONALISATION OF A MATRIX BY ORTHOGONAL TRANSFORMATION

Orthogonal matrix

Definition

A matrix 'A' is said to be orthogonal if $AA^T = A^TA = I$

Example: Show that the following matrix is orthogonal $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore$ A is orthogonal.

Modal Matrix

Modal matrix is a matrix in which each column specifies the eigenvectors of a matrix .It is denoted by N.

A square matrix A with linearity independent Eigen vectors can be diagonalized by a similarly transformation, $D = N^{-1}AN$, where N is the modal matrix .The diagonal matrix D has as its diagonal elements, the Eigen values of A.

Normalized vector

Eigen vector X_r is said to be normalized if each element of X_r is being divided by the square root of the sum of the squares of all the elements of X_r.i.e.,the normalized vector is $\frac{X}{|X|}$

$X_r = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; Normalized vector of $X_r = \begin{bmatrix} x_1/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_2/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_3/\sqrt{x_1^2 + x_2^2 + x_3^2} \end{bmatrix}$

Working rule for diagonalization of a square matrix A using orthogonal reduction:

i) Find all the Eigen values of the symmetric matrix A.

ii) Find the Eigen vectors corresponding to each Eigen value.

iii) Find the normalized modal matrix N having normalized Eigen vectors as its column vectors.

iv) Find the diagonal matrix $D = N^T AN$.The diagonal matrix D has Eigen values of A as its diagonal elements.

Example: Diagonalize the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{1} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 1 + 1 = 4$

$s_2 =$ sum of the minors of the main diagonalelement

$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$

$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$\Rightarrow \lambda = 1, (\lambda^2 - 3\lambda - 4) = 0$

$\Rightarrow \lambda = 1, \ (\lambda + 1)(\lambda - 4) = 0$

$\Rightarrow \lambda = -1, 1, 4$

To find the Eigen vectors:

Case (i) When $\lambda = 1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 - x_3 = 0 \ldots (1)$

$x_1 + 0x_2 - 2x_3 = 0 \ldots (2)$

$-x_1 - 2x_2 + 0x_3 = 0 \ldots (3)$

From (1) and (2)

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

Case(ii) When $\lambda = -1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = (-2 \quad 1 \quad -1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$X_2^T X_3 = (0 \quad 1 \quad 1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_3^T X_1 = (-1 \quad -1 \quad 1)\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} \mathbf{10} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{-2} & \mathbf{2} & \mathbf{3} \\ \mathbf{-5} & \mathbf{-3} & \mathbf{5} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = \text{sum of the main diagonal element}$

$= 10 + 2 + 5 = 17$

$s_2 = \text{sum of the minors of the main diagonalelement}$

$= \begin{vmatrix} 2 & 3 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 1 + 25 + 16 = 42$

$s_3 = |A| = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{vmatrix} = 0$

Characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 17\lambda + 42) = 0$

$\Rightarrow \lambda = 0, 3, 14$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-0 & -2 & -5 \\ -2 & 2-0 & 3 \\ -5 & -3 & 5-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$10x_1 - 2x_2 - 5x_3 = 0 \ldots (1)$

$-2x_1 + 2x_2 + 3x_3 = 0 \ldots (2)$

$-5x_1 + 3x_2 + 5x_3 = 0 \ldots (3)$

From (1) and (2)

$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$

$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$

$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$

$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

Case (ii) When $\lambda = 3$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-3 & -2 & -5 \\ -2 & 2-3 & 3 \\ -5 & -3 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \ldots (4)$$

$$-2x_1 - x_2 + 3x_3 = 0 \ldots (5)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 14$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-14 & -2 & -5 \\ -2 & 2-14 & 3 \\ -5 & -3 & 5-14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \ldots (7)$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \ldots (8)$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$; $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 5 + 4 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 1 + 2 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -3 - 5 + 8 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 6 + 3 + 3 = 12$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

To find the Eigen vectors:

Case (i) When $\lambda = 8$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \ldots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \dots (5)$$

$$2x_1 - x_2 + x_3 = 0 \dots (6)$$

Put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both X_1 and X_2

$(i.e)\ X_1^T X_3 = 0\ \&\ X_2^T X_3 = 0$

$$\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\ \&\ \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$2a - b + c = 0 \dots (7)$$

$$a + b + c = 0 \dots (8)$$

From (7) and (8)

$$\frac{a}{-1-1} = \frac{b}{0-2} = \frac{c}{2}$$

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^TAN$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 3 + 3 + 3 = 9$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 + 8 + 8 = 24$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 16$$

Characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 8\lambda + 16) = 0$$

$$\Rightarrow \lambda = 1, 4, 4$$

To find the Eigen vectors:

Case (i) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-1 & 1 & 1 \\ 1 & 3-1 & -1 \\ 1 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + x_2 + x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - x_3 = 0 \ldots (2)$$

$$x_1 - x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-4 & 1 & 1 \\ 1 & 3-4 & -1 \\ 1 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \ldots (4)$$

$$x_1 - x_2 - x_3 = 0 \ldots (5)$$

$$x_1 - x_2 - x_3 = 0 \ldots (6)$$

put $x_1 = 0 \Rightarrow x_2 = -x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both X_1 and X_2

$(i.e)$ $X_1^T X_3 = 0$ & $X_2^T X_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ \& \ \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-a + b + c = 0 \ldots (7)$$

$$0a - b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{1+1} = \frac{b}{0+1} = \frac{c}{1+0}$$

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$