

## INTRODUCTION

### Random Process:

Consider a random experiment with a sample space  $S$ . If a time function  $X(t, s)$  is assigned to each outcome  $s \in S$  and where  $t \in T$ , then the family of all such functions, denoted by  $\{X(t, s)\}$ , where  $s \in S$ ,  $t \in T$  is called a random process. In other words, a random process is a collection of random variables together with time.

**Note:** A random process is also called stochastic process.

### 3.1 Classification of Random Process:

Classify a random process according to the characteristic of  $T$  and the state space  $S$ . We shall consider only 4 cases based on  $T$  and  $S$ .

- i) Continuous random process
- ii) Continuous random sequence
- iii) Discrete random process
- iv) Discrete random sequence

### Continuous random Process:

If both  $S$  and  $T$  are continuous, then the random process is called continuous Random process.

### Continuous Random Sequence:

If  $S$  is continuous and  $T$  is discrete, then the random process is called continuous random sequence.

**Discrete Random Process:**

If S is discrete and T is continuous, then the random process is called discrete random process.

**Discrete Random Sequence:**

If both S and T are discrete, then the random process is called discrete random process.

**Deterministic Random Process:**

A random process is called a deterministic random process if all the future values are predicted from past observation.

**Non Deterministic Random Process:**

A random process is called a non - deterministic random process if the future values of any sample function cannot be predicted from the past observation.

**Wide Sense Stationary Process (WSS);**

A process  $\{X(t)\}$  is said to be Wide Sense Stationary Process if

$$(i) \text{Mean} = E[X(t)] = \text{constant}$$

$$(ii) \text{Auto correlation } R_{XX}(\tau) = E[X(t)X(t + \tau)] \text{ depends on } \tau$$

**Note:**

A WSS process is also called as Weak Sense Stationary Process.

A SSS process is also called a strongly stationary process.

For stationary process mean and variance are constants.

A random process, which is not stationary in any sense, is called evolutionary.

**Formulae:****Wide Sense Stationary (WSS):**

$$(i) \text{Mean} = E[X(t)] = \text{constant}$$

$$(ii) \text{Auto correlation } R_{XX}(\tau) = E[X(t)X(t + \tau)] \text{ depends on } \tau$$

**Stationary Process:**

$$(i) E[X(t)] = \text{constant}$$

$$(ii) \text{Var}[X(t)] = \text{constant}$$

**Strict Sense Stationary (SSS):**

$$E[X^n(t)] \text{ is a constant for every } n$$

**Joint Wide Sense Stationary (JWSS):**

$$(i) E[X(t)] = \text{constant}$$

$$(ii) E[Y(t)] = \text{constant}$$

$$(iii) R_{XX}(t, t + \tau) = E[X(t)Y(t + \tau)] \text{ depends on } \tau$$

**Mean Ergodic:**

$$\text{Time average, } \overline{X_T} = \frac{1}{2T} \int_{-T}^T X(t) dt$$

$$E[X(t)] = \lim_{T \rightarrow \infty} \overline{X_T}$$

**Correlation Ergodic:**

$$\overline{X_T} = \frac{1}{2T} \int_{-T}^T X(t) X(t + \tau) dt$$

$$R_{XX}(t, t + \tau) = \lim_{T \rightarrow \infty} \overline{X_T}$$

If  $Y(t) = X(t + a) - X(t - a)$ , prove that  $R_{YY}(\tau) = \langle 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a) \rangle$

**Solution:**

Given  $Y(t) = X(t + a) - X(t - a)$

$$R_{YY}(t) = E[Y(t_1)Y(t_2)]$$

$$= E[(X(t_1 + a) - X(t_1 - a))(X(t_2 + a) - X(t_2 - a))]$$

$$= E[(X(t_1 + a)X(t_2 + a) - X(t_1 + a)X(t_2 - a) - X(t_1 - a)X(t_2 + a) + X(t_1 - a)X(t_2 - a))]$$

$$= E[X(t_1 + a)X(t_2 + a)] - E[X(t_1 + a)X(t_2 - a)] - E[X(t_1 - a)X(t_2 + a)] + E[X(t_1 - a)X(t_2 - a)]$$

$$= R_{XX}(t_1 + a, t_2 + a) - R_{XX}(t_1 + a, t_2 - a) - R_{XX}(t_1 - a, t_2 + a) + R_{XX}(t_1 - a, t_2 - a)$$

$$= R_{XX}(t_1 + a - t_2 - a) - R_{XX}(t_1 + a - t_2 + a) - R_{XX}(t_1 - a - t_2 - a) + R_{XX}(t_1 - a - t_2 + a)$$

$$= R_{XX}(t_1 - t_2) - R_{XX}(t_1 - t_2 + 2a) - R_{XX}(t_1 - t_2 - 2a) + R_{XX}(t_1 - t_2)$$

$$= R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a) + R_{XX}(\tau)$$

$$R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$$

The following formulas are very useful to solve problems under stationary process.

> If  $X$  is a RV with mean zero, then  $\text{Var}(X) = E(X^2)$

$$> 1 + 2x + 3x^2 + \dots = (1 - x)^{-2}$$

$$> 1 + 4x + 9x^2 + \dots = (1 + x)(1 - x)^{-3}$$

> If  $A$  and  $B$  are RV's and  $\lambda$  is a constant, then

$$E[A\cos \lambda t + B\sin \lambda t] = E(A)\cos \lambda t + E(B)\sin \lambda t$$

$\therefore E(\cos \lambda \tau) = \cos \lambda \tau$ , since  $\lambda$  and  $\tau$  are constants.

### 3.2 STATIONARY PROCESS

#### Problems under Stationary process:

For a stationary process

(1)  $E[X(t)]$  is a constant

(2)  $\text{Var}[X(t)]$  is a constant

1. The process  $\{X(t)\}$  whose probability distribution under certain

conditions is given by  $P[X(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{(n+1)}}; n = 1, 2, 3, \dots \\ \frac{at}{1+at}; n = 0 \end{cases}$

Show that it is not a stationary process (Evolutionary).

**Solution:**

$n$	0	1	2	3	...
$p_n(t)$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	...

For a stationary process,

(1)  $E[X(t)]$  is a constant

(2)  $\text{Var}[X(t)]$  is a constant

$$\begin{aligned}
 E[X(t)] &= \sum_{n=0}^{\infty} np_n(t) = 0 + \frac{1}{(1+at)^2} + (2) \frac{at}{(1+at)^3} + (3) \frac{(at)^2}{(1+at)^4} \dots \\
 &= \frac{1}{(1+at)^2} \left[ 1 + 2 \frac{at}{1+at} + 3 \frac{(at)^2}{(1+at)^2} + \dots \dots \dots \right] \\
 &= \frac{1}{(1+at)^2} \left[ 1 - \frac{at}{(1+at)} \right]^{-2} \\
 &= \frac{1}{(1+at)^2} \left[ \frac{1+at-at}{(1+at)} \right]^{-2} \\
 &= \frac{1}{(1+at)^2} \left[ \frac{1}{(1+at)} \right]^{-2} \\
 &= \frac{1}{(1+at)^2} (1+at)^2 = 1
 \end{aligned}$$

$E[X(t)] = 1$  which is a constant

$$\begin{aligned}
 E[X^2(t)] &= \sum_{n=1}^{\infty} n^2 p_n(t) = 0 + \frac{1}{(1+at)^2} + (4) \frac{at}{(1+at)^3} + (9) \frac{(at)^2}{(1+at)^4} + \dots \\
 &= \frac{1}{(1+at)^2} \left[ 1 + 4 \frac{at}{1+at} + 9 \frac{(at)^2}{(1+at)^2} + \dots \right] \\
 &= \frac{1}{(1+at)^2} \left( 1 + \frac{at}{1+at} \right) \left[ 1 - \frac{at}{1+at} \right] \\
 &= \frac{1}{(1+at)^2} \left( \frac{1+2at}{1+at} \right) (1+at)^3
 \end{aligned}$$

$E[X^2(t)] = 1 + 2at$ , which is not a constant

$$\text{Var} [ X(t) ] = E[X^2(t)] - [E[X(t)]]^2 = 1 + 2at - 1$$

=  $2at$  which is not a constant.

$\therefore \{X(t)\}$  is not a stationary process.

**2. Consider a random process  $A_1$  and  $A_2$  are independent random variables with  $E(A_i) = a_i$  and  $\text{Var}(A_i) = \sigma_i^2$  for  $i = 1, 2$  Prove that the process  $\{X(t)\}$  is evolutionary.**

**Solution:**

Given  $X(t) = A_1 + A_2 t$  where  $A_1$  and  $A_2$  are independent random variables with  $E(A_i) = a_i$  and  $\text{Var}(A_i) = \sigma_i^2$  for  $i = 1, 2$

For a stationary process

(1)  $E[X(t)]$  is a constant

(2)  $\text{Var}[X(t)]$  is a constant

$$\begin{aligned} E[X(t)] &= E[A_1 + A_2 t] \\ &= E[A_1] + tE[A_2] \\ &= a_1 + ta_2 \end{aligned}$$

Which is not a constant.

Thus, the process  $\{X(t)\}$  is evolutionary.

**3. Let  $X(t) = B\sin \omega t$ , where  $B$  is a random variable with mean and variance 1 and  $\omega$  is a constant. Check whether  $\{X(t)\}$  is a stationary or not**

**Solution:**

Given  $X(t) = B\sin \omega t$ , where

$B$  is a random variable with Mean=0 and Variance =1

$$\text{Mean of } B = 0 \Rightarrow E(B) = 0 \dots\dots(i)$$

$$\text{Variance of } B = 1 \Rightarrow E(B^2) = 1 \dots\dots(ii)$$

For a stationary process,

$$(1) E[X(t)] \text{ is a constant}$$

$$(2) \text{Var}[X(t)] \text{ is a constant}$$

$$(1) E[X(t)] = E[B\sin \omega t]$$

$$= E[B]\sin \omega t$$

$$= 0 \text{ From (i)}$$

$\therefore E[X(t)]$  is a constant

$$(2) E[X^2(t)] = E[B^2\sin^2 \omega t]$$

$$= E(B^2)\sin^2 \omega t$$

$$= \sin^2 \omega t \text{ which is not a constant From (ii)}$$



$\text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]]^2 = \sin^2 \omega t$ , which is not a constant.

Since the condition (2) for Stationary Process is not satisfied,

Hence  $\{X(t)\}$  is not a Stationary Process.

**4. Consider the random process  $X(t) = \cos(t + \varphi)$  where  $\varphi$  is a random variable with density function  $f(\varphi) = \frac{1}{\pi}$ , where  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ . Check whether or not the process is stationary.**

**Solution:**

$X(t) = \cos(t + \varphi)$  where  $\varphi$  is a random variable with

$f(\varphi) = \frac{1}{\pi}$ , where  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$

For a stationary process,

(1)  $E[X(t)]$  is a constant

(2)  $\text{Var}[X(t)]$  is a constant

$E[X(t)] = E[\cos(t + \varphi)]$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) f(\varphi) d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) \frac{1}{\pi} d\varphi$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) d\varphi$$

$$= \frac{1}{\pi} [\sin(t + \varphi)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \left[ \sin \left( t + \frac{\pi}{2} \right) - \sin \left( t - \frac{\pi}{2} \right) \right]$$

$$= \frac{1}{\pi} \left[ \sin \left( \frac{\pi}{2} + t \right) + \sin \left( \frac{\pi}{2} - t \right) \right]$$

$$\because \sin \left( \frac{\pi}{2} - \theta \right) = \sin \left( \frac{\pi}{2} + \theta \right) = \cos \theta$$

$$= \frac{1}{\pi} [\cos t + \cos t]$$

$$= \frac{1}{\pi} 2 \cos t$$

$E[X(t)] = \frac{1}{\pi} 2 \cos t$ , which depends on  $t$ .

Since the condition (1) for Stationary Process is not satisfied,

$\{X(t)\}$  is not a Stationary Process.

**5. Let  $X(t) = \cos(\omega t + \theta)$ , where  $\theta$  is a random variable uniformly distributed over  $(0, 2\pi)$ . Prove that  $\{X(t)\}$  is a stationary process of first order.**

**Solution:**

Given:  $X(t) = \cos(\omega t + \theta)$ , where  $\theta$  is random variable uniformly distributed over  $(0, 2\pi)$ .

$$\therefore f_{\theta}(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

**To prove  $\{X(t)\}$  is a first order stationary process.**

we have to prove  $f_X(x; t)$  is independent of time.

**To find  $f_X(x; t)$  :**

We have  $x = \cos(\omega t + \theta)$

$$\Rightarrow \omega t + \theta = \pm \cos^{-1}[x]$$

**To find  $f_X(x; t)$ ,**

Take  $x = X(t)$

$$\Rightarrow \theta = -\omega t \pm \cos^{-1}[x] \quad \because \cos[\pm(\omega t + \theta)] = \cos(\omega t + \theta)$$

Let  $\theta_1 = -\omega t - \cos^{-1} x$  and  $\theta_2 = -\omega t + \cos^{-1} x$

$$\frac{d\theta_1}{dx} = 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\theta_2}{dx} = 0 + \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

The first order density of  $\{X(t)\}$  is given by

$$\begin{aligned} f_X(x, t) &= \left| \frac{d\theta_1}{dx} \right| f_{\theta}(\theta_1) + \left| \frac{d\theta_2}{dx} \right| f_{\theta}(\theta_2) \\ &= \left| \frac{1}{\sqrt{1-x^2}} \right| \frac{1}{2\pi} + \left| \frac{-1}{\sqrt{1-x^2}} \right| \frac{1}{2\pi} \end{aligned}$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\pi} \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2}{2\pi} \frac{1}{\sqrt{1-x^2}}$$

$$f_X(x, t) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$$

**To find the range of  $x$  :**

We have  $x = X(t) = \cos(\omega t + \theta)$ .

Since the value of  $\cos(\omega t + \theta)$  lies between  $-1$  and  $+1$ , we have  $-1 \leq x \leq 1$ .

$$\therefore f_X(x, t) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, -1 \leq x \leq 1 \dots \dots (1)$$

which is independent of time.

Hence,  $\{X(t)\}$  is a stationary process of first order.

**Problems on Wide Sense Stationary (WSS):**

**1. Show that the random process  $X(t) = A \cos(\omega t + \theta)$  is WSS, where  $A$  and  $\omega$  are constants and  $\theta$  is uniformly distributed on the interval  $(0, 2\pi)$**

**Solution:**

$$\text{Given, } X(t) = A \cos(\omega t + \theta)$$

$\theta$  is uniformly distributed on the interval  $(0, 2\pi)$

$$f(\theta) = \frac{1}{b-a}, a < \theta < b$$

$$f(\theta) = \frac{1}{2\pi}, 0 < \theta < 2\pi$$

**To Prove  $X(t)$  is WSS.**

(i) Mean =  $E[X(t)] = \text{constant}$

(ii) Auto correlation  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$  depends on  $\tau$

$$\begin{aligned} \text{(i)} \quad E[X(t)] &= \int_{-\infty}^{\infty} X(t)f(\theta)d\theta \\ &= \int_0^{2\pi} A \cos(\omega t + \theta) \frac{1}{2\pi} d\theta \\ &= \frac{A}{2\pi} [\sin(\omega t + \theta)]_0^{2\pi} \\ &= \frac{A}{2\pi} [\sin(\omega t + 2\pi) - \sin(\omega t + 0)] \\ &= \frac{A}{2\pi} [\sin \omega t - \sin \omega t] = 0 \end{aligned}$$

$E[X(t)] = 0$  is constant.

(ii)  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$

$$= E[A \cos(\omega t + \theta) A \cos(\omega(t + \tau) + \theta)]$$

$$= E[A^2]E[\cos(\omega t + \theta) \cos(\omega(t + \tau) + \theta)]$$

$$= A^2 \frac{1}{2} E[\cos(\omega t + \theta + \omega t + \omega \tau + \theta) \cos(\omega t + \theta - \omega t - \omega \tau - \theta)]$$

$$\cos(-\theta) = \cos \theta$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$= A^2 \frac{1}{2} E[\cos(2\omega t + 2\theta + \omega\tau) \cos(-\omega\tau)]$$

$$= \frac{A^2}{2} E[\cos(2\omega t + 2\theta + \omega\tau) \cos(\omega\tau)]$$

$$= \frac{A^2}{2} \left[ \cos \omega\tau + \int_0^{2\pi} \cos(2\omega t + 2\theta + \omega\tau) \frac{1}{2\pi} d\theta \right]$$

$$= \frac{A^2}{2} \cos \omega\tau + \frac{A^2}{4\pi} \left[ \frac{\sin(2\omega t + 2\theta + \omega\tau)}{2} \right]_0^{2\pi}$$

$$= \frac{A^2}{2} \cos \omega\tau + \frac{A^2}{8\pi} [\sin(2\omega t + \omega\tau + 4\pi) - \sin(2\omega t + \omega\tau)]$$

$$= \frac{A^2}{2} \cos \omega\tau + \frac{A^2}{8\pi} [\sin(2\omega t + \omega\tau) - \sin(2\omega t + \omega\tau)]$$

$$= \frac{A^2}{2} \cos \omega\tau + \frac{A^2}{8\pi} [0]$$

$$R_{XX}(\tau) = \frac{A^2}{2} \cos \omega\tau$$

Hence  $X(t)$  is WSS process.

**2. Show that the random process  $X(t) = A \cos \lambda t + B \sin \lambda t$  where  $\lambda$  is a constant,  $A$  and  $B$  are random variables, is WSS if (i)  $E[A] = E[B] = 0$  (ii)  $E[A^2] = E[B^2]$  and (iii)  $E[AB] = 0$**

**Solution:**

Given,  $X(t) = A \cos \lambda t + B \sin \lambda t$

$$E[A] = E[B] = 0, E[A^2] = E[B^2], E[AB] = 0$$

**To Prove  $X(t)$  is WSS.**

(i)  $Mean = E[X(t)] = constant$

(ii) *Auto correlation*  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$  depends on  $\tau$

(i)  $E[X(t)] = E[A \cos \lambda t + B \sin \lambda t]$

$$= E[A] \cos \lambda t + E[B] \sin \lambda t$$

$$= 0 * \cos \lambda t + 0 * \sin \lambda t$$

$$E[X(t)] = 0 \text{ is constant.}$$

(ii)  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$

$$= E[(A \cos \lambda t + B \sin \lambda t)(A \cos \lambda(t + \tau) + B \sin \lambda(t + \tau))]$$

$$= E[A^2 \cos \lambda t \cos \lambda(t + \tau) + AB \cos \lambda t \sin \lambda(t + \tau)$$

$$+ AB \sin \lambda t \cos \lambda(t + \tau) + B^2 \sin \lambda t \sin \lambda(t + \tau)]$$

$$= E[A^2 \cos \lambda t \cos \lambda(t + \tau)] + E[AB \cos \lambda t \sin \lambda(t + \tau)] +$$

$$E[AB \sin \lambda t \cos \lambda(t + \tau)] + E[B^2 \sin \lambda t \sin \lambda(t + \tau)]$$

$$= E[A^2 \cos \lambda t \cos \lambda(t + \tau)] + E[B^2 \sin \lambda t \sin \lambda(t + \tau)]$$

$$= E[A^2] \cos \lambda t \cos \lambda(t + \tau) + E[B^2] \sin \lambda t \sin \lambda(t + \tau)$$

$$= k \cos \lambda t \cos \lambda(t + \tau) + k \sin \lambda t \sin \lambda(t + \tau)$$

$$= k[\cos \lambda t \cos \lambda(t + \tau) + \sin \lambda t \sin \lambda(t + \tau)]$$

$$= k[\cos(\lambda t - \lambda t - \lambda\tau)]$$

$$\cos(-\theta) = \cos \theta$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

$$= k[\cos(-\lambda\tau)]$$

$$= k[\cos(\lambda\tau)]$$

Hence  $X(t)$  is WSS process.

**3. Given a random variable  $y$  with characteristic function  $\varphi(\omega) = E[e^{i\omega y}]$  and a random process  $X(t) = \cos(\lambda t + y)$ . Show that  $X(t)$  is stationary in the wide sense if  $\varphi(1) = \varphi(2) = 0$**

**Solution:**

Given,  $X(t) = \cos(\lambda t + y)$

$$\varphi(\omega) = E[e^{i\omega y}] = E[\cos \omega y + i \sin \omega y]$$

$$= E[\cos \omega y] + i E[\sin \omega y]$$

Given,  $\varphi(1) = 0$

$$\Rightarrow 0 = E[\cos y] + i E[\sin y]$$

$$E[\cos y] = 0; E[\sin y] = 0$$



Given,  $\varphi(2) = 0$

$$\Rightarrow 0 = E[\cos 2y] + i E[\sin 2y]$$

$$E[\cos 2y] = 0; E[\sin 2y] = 0$$

**To Prove  $X(t)$  is WSS.**

(i) Mean  $= E[X(t)] = \text{constant}$

(ii) Auto correlation  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$  depends on  $\tau$

(i)  $E[X(t)] = E[\cos(\lambda t + y)]$

$$= E[\cos \lambda t \cos y - \sin \lambda t \sin y]$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$= \cos \lambda t E[\cos y] - \sin \lambda t E[\sin y]$$

$$= \cos \lambda t * 0 - \sin \lambda t * 0$$

$E[X(t)] = 0$  is constant.

(ii)  $R_{XX}(\tau) = E[X(t)X(t + \tau)]$

$$= E[\cos(\lambda t + y) \cos(\lambda(t + \tau) + y)]$$

$$= E[\cos(\lambda t + y) \cos(\lambda t + \lambda \tau + y)]$$

$$= \frac{1}{2} E[\cos(\lambda t + y + \lambda t + \lambda \tau + y) + \cos(\lambda t + y - \lambda t - \lambda \tau - y)]$$

$$\begin{aligned}
&= \frac{1}{2} E[\cos(2\lambda t + 2y + \lambda\tau) + \cos(-\lambda\tau)] \\
&= \frac{1}{2} \cos \lambda\tau + \frac{1}{2} E[\cos(2\lambda t + \lambda\tau) \cos 2y - \sin(2\lambda t + \lambda\tau) \sin 2y] \\
&= \frac{1}{2} \cos \lambda\tau + \frac{1}{2} \cos(2\lambda t + \lambda\tau) E[\cos 2y] - \frac{1}{2} \sin(2\lambda t + \lambda\tau) E[\sin 2y] \\
&= \frac{1}{2} \cos \lambda\tau + \frac{1}{2} (0) \\
R_{XX}(\tau) &= \frac{1}{2} \cos \lambda\tau
\end{aligned}$$

Hence  $\{X(t)\}$  is WSS process.

**4. Show that the process  $X(t) = Y\cos \omega t + Z\sin \omega t$  where  $Y$  and  $Z$  independent RV's which follows  $N(0, \sigma^2)$  and  $\omega$  is a constant, is wide sense stationary.**

**Solution:**

Given  $X(t) = Y\cos \omega t + Z\sin \omega t$ , where  $Y$  and  $Z$  are independent

$$(i) E(Y) = E(Z) = 0$$

$$(ii) E(YZ) = 0$$

$$(iii) E(Y^2) = E(Z^2) = \sigma^2$$

**To prove  $\{X(t)\}$  is a WSS process,**

(1)  $E[X(t)]$  is a constant

(2)  $R_{XX}(t_1, t_2)$  is a function of  $\tau$

$$(1) E[X(t)] = E[Y \cos \omega t + Z \sin \omega t]$$

$$= E(Y) \cos \omega t + E(Z) \sin \omega t$$

$$= 0 + 0 = 0 \text{ From (i)}$$

$\therefore E[X(t)]$  is a constant

$$(2) R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[(Y \cos \omega t_1 + Z \sin \omega t_1)(Y \cos \omega t_2 + Z \sin \omega t_2)]$$

$$= E[Y^2 \cos \omega t_1 \cos \omega t_2 + YZ \sin \omega t_2 \cos \omega t_1 + ZY \sin \omega t_1 \cos \omega t_2 + Z^2 \sin \omega t_1 \sin \omega t_2]$$

$$= E(Y^2) \cos \omega t_1 \cos \omega t_2 + E(YZ) \sin \omega t_2 \cos \omega t_1 + E(YZ) \sin \omega t_1 \cos \omega t_2 + E(Z^2) \sin \omega t_1 \sin \omega t_2$$

$$= \sigma^2 \cos \omega t_1 \cos \omega t_2 + 0 + 0 +$$

$$\sigma^2 \sin \omega t_1 \sin \omega t_2 \text{ From (ii) \& (iii)}$$

$$= \sigma^2 [\cos \omega t_1 \cos \omega t_2 + \sin \omega t_1 \sin \omega t_2]$$

$$= \sigma^2 \cos(\omega t_1 - \omega t_2)$$

$$= \sigma^2 \cos[\omega(t_1 - t_2)]$$

$$R_{XX}(t_1, t_2) = \sigma^2 \cos \omega \tau$$

$R_{XX}(t_1, t_2)$  is a function of  $\tau$ .

Since the conditions (1) and (2) for WSS are satisfied.

Hence,  $\{X(t)\}$  is a WSS Process.

**5. If  $X(t) = Y\cos t + Z\sin t$ , where  $Y$  and  $Z$  are independent binary random variables each of which assumes the values  $-1$  and  $+2$  with probabilities  $\frac{2}{3}$  and  $\frac{1}{3}$  respectively. Prove that  $\{X(t)\}$  is WSS.**

**Solution:**

Given  $X(t) = Y\cos t + Z\sin t$ , where  $Y$  and  $Z$  are independent binary random variables. The probability distribution of  $Y$  and  $Z$  are given by

y	-1	2	z	-1	2
P(y)	2/3	1/3	P(z)	2/3	1/3

$$E(Y) = \sum yp(y) = (-1)\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right)$$

$$E(Y) = 0$$

$$E(Y^2) = \sum y^2p(y) = 1\left(\frac{2}{3}\right) + 4\left(\frac{1}{3}\right)$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \frac{6}{3} = 2$$

Similarly,  $E(Z) = 0, E(Z^2) = 2$ .

Since,  $Y$  and  $Z$  are independent,  $E(YZ) = E(Y)E(Z) = 0$

**To prove  $\{X(t)\}$  is a WSS process**

(1)  $E[X(t)]$  is a constant

(2)  $R_{XX}(t_1, t_2)$  is a function of  $n$  of  $\tau$

$$\begin{aligned} (1) E[X(t)] &= E[Y\cos t + Z\sin t] \\ &= E(Y)\cos t + E(Z)\sin t = 0 \end{aligned}$$

$\therefore E[X(t)]$  is a constant.

$$\begin{aligned} (2) R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[(Y\cos t_1 + Z\sin t_1)(Y\cos t_2 + Z\sin t_2)] \\ &= E[Y^2\cos t_1\cos t_2 + YZ\sin t_2\cos t_1 + YZ\sin t_1\cos t_2 + Z^2\sin t_1\sin t_2] \\ &= E(Y^2)\cos t_1\cos t_2 + E(YZ)\sin t_2\cos t_1 + E(YZ)\sin t_1\cos t_2 \\ &\quad + E(Z^2)\sin t_1\sin t_2 \\ &= 2\cos t_1\cos t_2 + 2\sin t_1\sin t_2 \\ &= 2\cos(t_1 - t_2) = 2\cos \tau \end{aligned}$$

$R_{XX}(t_1, t_2)$  is a function of  $\tau$ .

Since the conditions (1) and (2) for WSS are satisfied.

Hence,  $\{X(t)\}$  is a WSS Process.

6. Consider the process  $X(t) = \sum_{i=1}^n (A_i \cos p_i t + B_i \sin p_i t)$  where  $A_i$  and  $B_i$  are uncorrelated R.v's with mean '0' & variance  $\sigma_i^2$ . Prove that  $\{X(t)\}$  is a WSS process.

**Solution:**

$$\text{Given } X(t) = \sum_{i=1}^n (A_i \cos p_i t + B_i \sin p_i t)$$

$A_i$  and  $B_i$  are Rv's

Given Means of  $A_i$  and  $B_i = 0 \Rightarrow E[A_i] = 0$  &  $E[B_i] = 0$  and

$$\text{Var}[A_i] = \text{Var}[B_i] = \sigma_i^2$$

$$\Rightarrow E[A_i^2] = E[B_i^2] = \sigma_i^2$$

Also  $A_i$  and  $B_i$  are uncorrelated  $\therefore E[A_i B_j] = 0$  for all  $i, j$

$$\therefore E[A_i A_j] = E[B_i B_j] = 0 \text{ for } i \neq j$$

**To prove  $\{X(t)\}$  is a WSS process**

(1)  $E[X(t)]$  is a constant

(2)  $R_{XX}(t_1, t_2)$  is a function of  $\tau$ .

$$(1) E[X(t)] = E\left[\sum_{i=1}^n (A_i \cos p_i t + B_i \sin p_i t)\right]$$

$$= \sum_{i=1}^n [E(A_i) \cos p_i t + E(B_i) \sin p_i t] = 0$$

$\therefore E[X(t)]$  is a constant.

2) The ACF of  $\{X(t)\}$  is given by  $R(\tau) = E[X(t_1)X(t_2)]$

$$\begin{aligned}
 &= E\left[\sum_{i=1}^n (A_i \cos p_i t_1 + B_i \sin p_i t_1) \sum_{j=1}^n (A_j \cos p_j t_2 + B_j \sin p_j t_2)\right] \\
 &= E\left[\sum_{i=1}^n \sum_{j=1}^n (A_i \cos p_i t_1 + B_i \sin p_i t_1)(A_j \cos p_j t_2 + B_j \sin p_j t_2)\right] \\
 &= E\left[\sum_{i=1}^n \sum_{j=1}^n \left[ A_i A_j \cos p_i t_1 \cos p_j t_2 + A_i B_j \cos p_i t_1 \sin p_j t_2 + \right. \right. \\
 & \quad \left. \left. A_j B_i \sin p_i t_1 \cos p_j t_2 + B_i B_j \sin p_i t_1 \sin p_j t_2 \right)\right] \\
 &= \sum_{i=1}^n \sum_{j=1}^n \left[ E(A_i A_j) \cos p_i t_1 \cos p_j t_2 + E(A_i B_j) \cos p_i t_1 \sin p_j t_2 + \right. \\
 & \quad \left. E(A_j B_i) \sin p_i t_1 \cos p_j t_2 + E(B_i B_j) \sin p_i t_1 \sin p_j t_2 \right] \\
 &= \sum_{i=1}^n \sum_{j=1}^n [E(A_i A_j) \cos p_i t_1 \cos p_j t_2 + E(B_i B_j) \sin p_i t_1 \sin p_j t_2] \\
 &= \sum_{i=1}^n [E(A_i)^2 \cos p_i t_1 \cos p_i t_2 + E(B_i)^2 \sin p_i t_1 \sin p_i t_2] \\
 &= \sum_{i=1}^n \sigma_i^2 (\cos p_i t_1 \cos p_i t_2 + \sin p_i t_1 \sin p_i t_2) = \sum_{i=1}^n \sigma_i^2 \cos(p_i t_1 - p_i t_2) \\
 R(\tau) &= \sum_{i=1}^n \sigma_i^2 \cos p_i \tau
 \end{aligned}$$

$R_{XX}(t_1, t_2)$  is a function of  $\tau$ .

Since the conditions (1) and (2) for WSS are satisfied.

Hence,  $\{X(t)\}$  is a WSS Process.

**7. Let  $X(t) = B \sin(100t + \theta)$ , where  $B$  and  $\theta$  are independent  $RV$ 's such that  $\theta$  is uniform distributed over  $(-\pi, \pi)$  and  $B$  has mean '0' and variance**

‘1’. Find mean and auto correlation function of  $\{X(t)\}$

**Solution:**

Given  $X(t) = B\sin(100t + \theta)$ , where  $B$  and  $\theta$  are independent RV' s.

$\theta$  is uniform distributed over  $(-\pi, \pi)$ .

$$f(\theta) = \frac{1}{2\pi}, -\pi < \theta < \pi$$

Mean of  $B = 0 \Rightarrow E[B] = 0$

variance of  $B = 1 \Rightarrow E[B^2] = 1$

The mean of  $X(t)$  is given by

$$\begin{aligned} E[X(t)] &= E[B \sin(100t + \theta)] \\ &= E[B]E[\sin(100t + \theta)] \\ &= 0 \times E[\sin(100t + \theta)] \end{aligned}$$

$$E[X(t)] = 0$$

Mean of  $X(t) = 0$

The ACF of  $\{X(t)\}$  is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[B\sin(100t_1 + \theta)B\sin(100t_2 + \theta)] \\ &= E[B^2\sin(100t_1 + \theta)\sin(100t_2 + \theta)] \\ &= E[B^2]E[\sin(100t_1 + \theta)\sin(100t_2 + \theta)] \\ &= 1 \times \frac{1}{2}E[\cos(100t_1 + \theta - 100t_2 - \theta) - \cos(100t_1 + \theta + 100t_2 + \theta)] \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{2} [E[\cos(100t_1 - 100t_2) - \cos(100t_1 + 100t_2 + 2\theta)]] \\
 &= \frac{1}{2} [E(\cos 100\tau - \cos(100t_1 + 100t_2 + 2\theta))] \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{2} E[\cos(100t_1 + 100t_2 + 2\theta)] \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{2} \int_{-\pi}^{\pi} \cos(100t_1 + 100t_2 + 2\theta) f(\theta) d\theta \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{2} \int_{-\pi}^{\pi} \cos(100t_1 + 100t_2 + 2\theta) \frac{1}{2\pi} d\theta \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{4\pi} \left[ \frac{\sin(100t_1 + 100t_2 + 2\theta)}{2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{8\pi} [\sin(100t_1 + 100t_2 + 2\pi) - \sin(100t_1 + 100t_2 + 2\pi)] \\
 &= \frac{1}{2} \cos 100\tau - \frac{1}{8\pi} (0) = \frac{1}{2} \cos 100\tau
 \end{aligned}$$

Auto Correlation function of  $X(t) = \frac{1}{2} \cos 100\tau$

**8. Let  $X(t) = A \cos \lambda t + B \sin \lambda t$ , where  $\lambda$  is a constant and  $A$  &  $B$  are independent RV 's with mean '0' & variance 1. Prove that  $\{X(t)\}$  is covariance stationary.**

**Solution:**

Given  $X(t) = A \cos \lambda t + B \sin \lambda t$ , where  $A$  &  $B$  are RV's and  $\lambda$  is a constant.

Given  $E(A) = E(B) = 0$ .

Also given  $A$  and  $B$  are independent RV's.

$$\therefore E[AB] = E[A]E[B] = 0$$

$$\text{Also given } \text{Var}(A) = \text{Var}(B) = 1$$

$$\Rightarrow E[A^2] = E[B^2] = 1$$

**To prove  $\{X(t)\}$  is a covariance stationary process**

(1)  $E[X(t)]$  is a constant

(2)  $C_{XX}(t_1, t_2)$  is a function of  $\tau$

$$\begin{aligned} (1) E[X(t)] &= E[A \cos \lambda t + B \sin \lambda t] \\ &= E[A] \cos \lambda t + E[B] \sin \lambda t \end{aligned}$$

$$E[X(t)] = 0$$

$\therefore E[X(t)]$  is a constant.

$$\begin{aligned} (2) C_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)] \\ &= E[(A \cos \lambda t_1 + B \sin \lambda t_1)(A \cos \lambda t_2 + B \sin \lambda t_2)] - 0 \times 0 \\ &= E[A^2 \cos \lambda t_1 \cos \lambda t_2 + AB \cos \lambda t_1 \sin \lambda t_2 + \\ &\quad AB \sin \lambda t_1 \cos \lambda t_2 + B^2 \sin \lambda t_1 \sin \lambda t_2] \end{aligned}$$

$$\begin{aligned} &= E[A^2] \cos \lambda t_1 \cos \lambda t_2 + E[AB] \cos \lambda t_1 \sin \lambda t_2 + E[AB] \sin \lambda t_1 \cos \lambda t_2 + \\ &\quad E[B^2] \sin \lambda t_1 \sin \lambda t_2 \end{aligned}$$

$$= \cos \lambda t_1 \cos \lambda t_2 + 0 + 0 + \sin \lambda t_1 \sin \lambda t_2$$

$$= \cos \lambda(t_1 - t_2)$$

$$C_{XX}(t_1, t_2) = \cos \lambda \tau$$

$\therefore C_{XX}(t_1, t_2)$  is a function of  $\tau$

Since the conditions (1) and (2) for Covariance Stationary Process are  $\{X(t)\}$  is a covariance stationary process.

### Problems under SSS process:

For a SSS process,  $E[X^n(t)]$  is a constant for every  $n$ .

1. Verify whether the sine wave  $X(t) = Y \cos \omega t$ , where  $Y$  is a random variable uniformly distributed over  $(0, 1)$  is a SSS process or not.

### Solution:

Given,  $X(t) = Y \cos \omega t$ , where  $Y$  is a random variable uniformly distributed over  $(0, 1)$

$$\therefore f_Y(y) = 1; 0 < y < 1$$

For a SSS process,  $E[X^n(t)]$  is a constant for every  $n$ .

$$\begin{aligned} E[Y] &= \int_0^1 y f_Y(y) dy \\ &= \int_0^1 y dy = \left[ \frac{y^2}{2} \right]_0^1 = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E[X(t)] &= E[Y \cos \omega t] \\ &= E[Y] \cos \omega t \end{aligned}$$

$$= \frac{1}{2} \cos \omega t$$

$E[X(t)]$  is not a constant.

Hence  $\{X(t)\}$  is not a SSS process.

**2. Consider random process  $X(t)$  defined by  $X(t) = U \cos t + V \sin t$ , where  $U$  and  $V$  are independent random variables each of which assumes the values  $-2$  and  $1$  with probabilities  $1/3$  and  $2/3$  respectively. Show that  $\{X(t)\}$  is a wide sense stationary process and not a strict sense stationary process (SSS)**

**Solution:**

Given  $X(t) = U \cos t + V \sin t$  where  $U$  and  $V$  are R.V'S with the following probability distributions

u	-2	1
P(u)	1/3	2/3

v	-2	1
P(v)	1/3	2/3

$$E(U) = \sum up(u) = \left(-2 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = -\frac{2}{3} + \frac{2}{3} = 0$$

$$E(U^2) = \sum u^2 p(u) = \left(4 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = \frac{4}{3} + \frac{2}{3} = 2$$

$$E(U^3) = \sum u^3 p(u) = \left(-8 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right) = -\frac{8}{3} + \frac{2}{3} = -2$$

Similarly  $E(V) = 0, E(V^2) = 2, E(V^3) = -2$

Since  $U$  and  $V$  are independent R.V'S, follow that

$$E(UV) = E(U)E(V) = 0 \times 0 = 0$$

$$E(U^2V) = E(U^2)E(V) = 2 \times 0 = 0$$

$$E(UV^2) = E(U)E(V^2) = 0 \times 2 = 0$$

**To Prove  $X(t)$  is WSS.**

(i)  $Mean = E[X(t)] = constant$

(ii) *Auto correlation*  $R_{XX}(\tau) = E[X(t_1)X(t_2)]$  depends on  $\tau$

(i)  $E[X(t)] = E[U \cos t + V \sin t]$

$$= E[U] \cos t + E[V] \sin t$$

$$= 0 + 0 = 0$$

$E[X(t)]$  is a constant.

(ii)  $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

$$= E[(U \cos t_1 + V \sin t_1)(U \cos t_2 + V \sin t_2)]$$

$$= E[U^2 \cos t_1 \cos t_2 + UV \cos t_1 \sin t_2 + UV \sin t_1 \cos t_2 + V^2 \sin t_1 \sin t_2]$$

$$= E[U^2 \cos t_1 \cos t_2] + E[UV \cos t_1 \sin t_2] + E[UV \sin t_1 \cos t_2] +$$

$$E[V^2 \sin t_1 \sin t_2]$$

$$= E[U^2 \cos t_1 \cos t_2] + E[V^2 \sin t_1 \sin t_2]$$

$$= E[U^2] \cos t_1 \cos t_2 + E[V^2] \sin t_1 \sin t_2$$

$$= 2 \cos t_1 \cos t_2 + 2 \sin t_1 \sin t_2$$

$$= 2[\cos t_1 \cos t_2 + \sin t_1 \sin t_2]$$

$$= 2[\cos(t_1 - t_2)]$$

$$\cos(-\theta) = \cos \theta$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

$$= 2[\cos \tau]$$

Since the conditions (1) and (2) for WSS are satisfied,  $X(t)$  is WSS process.

**To check  $\{X(t)\}$  is Strict Sense Stationary**

$$E[X^3(t)] = E[(U \cos t + V \sin t)^3]$$

$$= E(U^3 \cos^3 t + 3U^2 V \cos^2 t \sin t + 3UV^2 \cos t \sin^2 t + V^3 \sin^3 t)$$

$$= E(U^3) \cos^3 t + 3E(U^2 V) \cos^2 t \sin t + 3E(UV^2) \cos t \sin^2 t + E(V^3) \sin^3 t$$

$$= -2 \cos^3 t + 0 + 0 - 2 \sin^3 t = -2(\cos^3 t + \sin^3 t)$$

Which depends on  $t$ .

Hence  $\{X(t)\}$  is not a strict sense stationary process.

**Cross Correlation Function:**

Let  $\{X(t)\}$  and  $\{Y(t)\}$  be two random processes. Then cross correlation function of  $X(t)$  and  $Y(t)$  is  $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

**Jointly WSS Process:**

Let  $\{X(t)\}$  and  $\{Y(t)\}$  be two random processes. Then the function of  $X(t)$  and  $Y(t)$  is said to be JWSS process if

$$(i) E[X(t)] = \text{constant}$$

$$(ii) E[Y(t)] = \text{constant}$$

$$(iii) R_{XY}(t_1, t_2) \text{ is a function of } \tau$$

**Problems under Joint Wide Sense Stationary:**

1. If  $X(t) = 5 \cos(10t + \theta)$  and  $Y(t) = 20 \sin(10t + \theta)$ , where  $\theta$  is uniformly distributed over  $(0, 2\pi)$ . Prove  $\{X(t)\}$  and  $\{Y(t)\}$  are JWSS.

**Solution:**

Given  $X(t) = 5 \cos(10t + \theta)$ ,  $Y(t) = 20 \sin(10t + \theta)$ , where  $\theta$  is a RV uniformly distributed over  $(0, 2\pi)$

$$f(\theta) = \frac{1}{2\pi}, 0 < \theta < 2\pi$$

To prove  $\{X(t)\}$  and  $\{Y(t)\}$  is a JWSS process.

$$(i) E[X(t)] = \text{constant}$$

$$(ii) E[Y(t)] = \text{constant}$$

(iii)  $R_{XY}(t, t + \tau)$  is a function of  $\tau$

$$(i) E[X(t)] = E[5 \cos(10t + \theta)]$$

$$= 5 E[\cos(10t + \theta)]$$

$$= 5 \int_0^{2\pi} \cos(10t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{5}{2\pi} \int_0^{2\pi} \cos(10t + \theta) d\theta$$

$$= \frac{5}{2\pi} [\sin(10t + \theta)]_0^{2\pi}$$

$$= \frac{5}{2\pi} [\sin(10t + 2\pi) - \sin(10t)]$$

$$= \frac{5}{2\pi} [\sin 10t - \sin 10t] = 0$$

$E[X(t)] = 0$  is a constant

$$(ii) E[Y(t)] = E[20 \sin(10t + \theta)]$$

$$= 20 E[\sin(10t + \theta)]$$

$$= 20 \int_0^{2\pi} \sin(10t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{20}{2\pi} \int_0^{2\pi} \sin(10t + \theta) d\theta$$

$$= \frac{10}{\pi} [-\cos(10t + \theta)]_0^{2\pi}$$

$$= \frac{10}{\pi} [-\cos(10t + 2\pi) + \cos(10t)]$$



$$= \frac{10}{\pi} [-\cos 10t + \cos 10t] = 0$$

$E[Y(t)] = 0$  is a constant

$$(iii) R_{XY}(\tau) = E[X(t)Y(t + \tau)]$$

$$= E[5 \cos(10t + \theta) 20 \sin(10(t + \tau) + \theta)]$$

$$= E[5 \cos(10t + \theta) 20 \sin(10t + 10\tau + \theta)]$$

$$= \frac{100}{2} E[\sin(10t + \theta + 10t + 10\tau + \theta) - \sin(10t + \theta - 10t - 10\tau - \theta)]$$

$$= 50E[\sin(20t + 10\tau + 2\theta) - \sin(-10\tau)]$$

$$= 50E[\sin(20t + 10\tau + 2\theta) + \sin(10\tau)]$$

$$= 50 \sin 10\tau + \frac{50}{2\pi} \int_0^{2\pi} \sin(20t + 10\tau + 2\theta) d\theta$$

$$= 50 \sin 10\tau + \frac{25}{\pi} \left[ \frac{-\cos(20t + 10\tau + 2\theta)}{2} \right]_0^{2\pi}$$

$$= 50 \sin 10\tau + \frac{25}{2\pi} [-\cos(20t + 10\tau + 4\pi) + \cos(20t + 10\tau)]$$

$$= 50 \sin 10\tau + \frac{25}{2\pi} [-\cos(20t + 10\tau) + \cos(20t + 10\tau)]$$

$$= 50 \sin 10\tau$$

$R_{XX}(t, t + \tau)$  is a function of  $\tau$

Since the conditions (i), (ii) and (iii) for JWSS are satisfied,  $\{X(t)\}$  and  $\{Y(t)\}$  are JWSS processes.

**2. Two random processes are obtained by  $X(t) = A \cos\omega_0 t + B \sin\omega_0 t$  and  $Y(t) = B \cos\omega_0 t - A \sin\omega_0 t$ . Show that  $X(t)$  and  $Y(t)$  are JWSS if  $A$  and  $B$  are uncorrelated random variables with zero mean and same variances and  $\omega_0$  is a constant.**

**Solution:**

$$\text{Given } X(t) = A \cos\omega_0 t + B \sin\omega_0 t \text{ and } Y(t) = B \cos\omega_0 t - A \sin\omega_0 t$$

Where  $A$  and  $B$  are random variables with mean zero.

$$\therefore E(A) = E(B) = 0 \dots\dots\dots(1)$$

Also given  $A$  and  $B$  have same variances

$$(i.e) \text{Var}(A) = \text{Var}(B)$$

$$\therefore E(A^2) = E(B^2) = 0 \dots\dots\dots(2)$$

Also given  $A$  and  $B$  are uncorrelated

$$E(AB) = E(A) E(B) = 0 \dots\dots\dots(3)$$

**To Prove  $\{X(t)\}$  and  $\{Y(t)\}$  is a JWSS process.**

$$(i) E[X(t)] = \text{constant}$$

$$(ii) E[Y(t)] = \text{constant}$$

(iii)  $R_{XY}(t_1, t_2)$  is a function of  $\tau$

$$\begin{aligned}
 \text{(i) } E[X(t)] &= E[A \cos \omega_0 t + B \sin \omega_0 t] \\
 &= E[A] \cos \omega_0 t + E[B] \sin \omega_0 t \\
 &= 0 \text{ (by 1)}
 \end{aligned}$$

$E[X(t)] = 0$  is a constant

$$\begin{aligned}
 \text{(ii) } E[Y(t)] &= E[B \cos \omega_0 t - A \sin \omega_0 t] \\
 &= E[B] \cos \omega_0 t - E[A] \sin \omega_0 t \\
 &= 0 \text{ (by 1)}
 \end{aligned}$$

$E[Y(t)] = 0$  is a constant

$$\begin{aligned}
 \text{(iii) } R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\
 &= E[(A \cos \omega_0 t_1 + B \sin \omega_0 t_1)(B \cos \omega_0 t_2 - A \sin \omega_0 t_2)] \\
 &= E[AB \cos \omega_0 t_1 \cos \omega_0 t_2 - A^2 \cos \omega_0 t_1 \sin \omega_0 t_2 + B^2 \sin \omega_0 t_1 \cos \omega_0 t_2 - \\
 &AB \sin \omega_0 t_1 \sin \omega_0 t_2] \\
 &= E[AB] \cos \omega_0 t_1 \cos \omega_0 t_2 - E[A^2] \cos \omega_0 t_1 \sin \omega_0 t_2 + \\
 &E[B^2] \sin \omega_0 t_1 \cos \omega_0 t_2 - E[AB] \sin \omega_0 t_1 \sin \omega_0 t_2] \\
 &= 0 - E[A^2] \cos \omega_0 t_1 \sin \omega_0 t_2 + E[B^2] \sin \omega_0 t_1 \cos \omega_0 t_2 - 0 \\
 &= -E[A^2] \cos \omega_0 t_1 \sin \omega_0 t_2 + E[B^2] \sin \omega_0 t_1 \cos \omega_0 t_2 \\
 &= E[A^2](\sin \omega_0 t_1 \cos \omega_0 t_2 - \cos \omega_0 t_1 \sin \omega_0 t_2) \\
 &= E[A^2] \sin \omega_0 (t_1 - t_2)
 \end{aligned}$$

$= E[A^2] \sin \omega_0 \tau$ , which is a function of  $\tau$

$\therefore R_{XY}(t_1, t_2)$  is a function of  $\tau$

$$\sin A \cos B - \cos A \sin B = \sin(A - B)$$

Since the conditions (1), (2) and (3) for JWSS are satisfied,  $X(t)$  and  $Y(t)$  are is JWSS process.

**Problems under Ergodic Process:**

1. Let  $X(t) = A$ , where  $A$  is a random variable. Prove that  $\{X(t)\}$  is not a mean ergodic.

**Solution:**

Given  $X(t) = A$ , where  $A$  is a random variable

To Prove  $\{X(t)\}$  is a mean ergodic, we have to prove

$$E[X(t)] = \lim_{T \rightarrow \infty} \overline{X_T}$$

The ensemble mean of  $\{X(t)\}$  is given by,

$$E[X(t)] = E[A] \text{ --- (1)}$$

The time average is given by,

$$\begin{aligned}\bar{X}_T &= \frac{1}{2T} \int_{-T}^T X(t) dt = \frac{1}{2T} \int_{-T}^T A dt \\ &= \frac{A}{2T} \int_{-T}^T dt = \frac{A}{2T} [t]_{-T}^T = \frac{A}{2T} (2T) = A\end{aligned}$$

$$\therefore \bar{X}_T = A$$

$$\lim_{T \rightarrow \infty} \bar{X}_T = A \text{ --- (2)}$$

From (1) and (2)

$$E[X(t)] \neq \lim_{T \rightarrow \infty} \bar{X}_T$$

$\therefore \{X(t)\}$  is not mean Ergodic.

**2. A random process has sample functions of the form  $X(t) = A \cos(\omega t + \theta)$ , where  $\omega$  is constant and  $A$  is a random variable with mean zero and variance one and  $\theta$  is also a random variable that is uniformly distributed between 0 and  $2\pi$ . Assume that the random variables  $A$  and  $\theta$  are independent. Prove that  $X(t)$  is a mean ergodic process?**

**Solution:**

Given  $X(t) = A \cos(\omega t + \theta)$ , where  $A$  is a random variable with mean zero .

$$\therefore E(A) = 0, E(A^2) = 1$$

$\theta$  is uniformly distributed between 0 and  $2\pi$

$$f(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

**To Prove {X(t)} is Mean Ergodic.**

we have to prove

$$E[X(t)] = \lim_{T \rightarrow \infty} \overline{X_T}$$

The ensemble mean of {X(t)} is given by,

$$\begin{aligned} E[X(t)] &= E[A \cos(\omega t + \theta)] \\ &= E[A] \cos(\omega t + \theta) \text{ since } A \text{ and } \theta \text{ are independent R.V'S} \\ &= 0 \dots \dots \dots (1) \end{aligned}$$

The time average is given by,

$$\begin{aligned} \overline{X_T} &= \frac{1}{2T} \int_{-T}^T X(t) dt \\ &= \frac{1}{2T} \int_{-T}^T A \cos(\omega t + \theta) dt \\ &= \frac{A}{2T} \int_{-T}^T \cos(\omega t + \theta) dt \\ &= \frac{A}{2T} \left[ \frac{\sin(\omega t + \theta)}{\omega} \right]_{-T}^T \\ \overline{X_T} &= \frac{A}{2T\omega} [\sin(\omega T + \theta) - \sin(-\omega T + \theta)] \\ \lim_{T \rightarrow \infty} \overline{X_T} &= \lim_{T \rightarrow \infty} \frac{A}{2T\omega} [\sin(\omega T + \theta) - \sin(-\omega T + \theta)] \\ &= 0 \dots \dots \dots (2) \end{aligned}$$

From (1) and (2) ,  $E[X(t)] = \lim_{T \rightarrow \infty} \overline{X_T}$

$\therefore \{X(t)\}$  is a mean Ergodic Process.

### Correlation Ergodic Process:

Let  $\{X(t)\}$  be a random process. The ensemble auto correlation function is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

The time auto correlation function is  $\overline{X_T} = \frac{1}{2T} \int_{-T}^T X(t) X(t + \tau) dt$

A process  $\{X(t)\}$  is said to be correlation ergodic if  $R_{XX}(t_1, t_2) = \lim_{T \rightarrow \infty} \overline{X_T}$

### Problems under Correlation Ergodic Process:

1. Given a WSS random process  $\{X(t)\} = 10 \cos(100t + \theta)$ , where  $\theta$  is uniformly distributed over  $(-\pi, \pi)$ . Prove that  $X(t)$  is a correlation ergodic.

### Solution:

$$\text{Given } \{X(t)\} = 10 \cos(100t + \theta)$$

$$\Rightarrow f(\theta) = \frac{1}{2\pi}, -\pi < \theta < \pi$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$\begin{aligned}
 &= E[10 \cos(100t + \theta) 10 \cos(100(t + \tau) + \theta)] \\
 &= \frac{100}{2} E[\cos(100t + \theta) \cos(100t + 100\tau + \theta)] \\
 &= 50 E[\cos(100t + \theta + 100t + 100\tau + \theta) + \cos(100t + \theta - 100t - 100\tau - \theta)] \\
 &= 50 E[\cos(200t + 2\theta + 100\tau) + \cos(-100\tau)] \\
 &= 50 E[\cos(200t + 2\theta + 100\tau) + \cos(100\tau)] \\
 &= 50 \cos(100\tau) + \frac{50}{2\pi} \int_0^{2\pi} \cos(200t + 100\tau + 2\theta) d\theta \\
 &= 50 \cos(100\tau) + \frac{25}{\pi} \left[ \frac{\sin(200t + 100\tau + 2\theta)}{2} \right]_0^{2\pi} \\
 &= 50 \cos 100\tau + \frac{25}{2\pi} [\sin(200t + 100\tau + 4\pi) - \sin(200t + 100\tau - 0)] \\
 &= 50 \cos 100\tau + \frac{25}{2\pi} [\sin(200t + 100\tau) - \sin(200t + 100\tau)] \\
 &R_{XX}(t, t + \tau) = 50 \cos 100\tau
 \end{aligned}$$

$$\text{Let } \overline{X_T} = \frac{1}{2T} \int_{-T}^T X(t) X(t + \tau) dt$$

$$\begin{aligned}
 &= \frac{1}{2T} \left[ \int_{-T}^T 50 \cos(100\tau) dt + 50 \int_{-T}^T \cos(200t + 100\tau + 2\theta) dt \right] \\
 &= \frac{1}{2T} \left\{ [50 \cos(100\tau) t]_{-T}^T + 50 \left[ \frac{\sin(200t + 100\tau + 2\theta)}{200} \right]_{-T}^T \right\}
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{2T} \left\{ [50 \cos(100\tau) (T - (-T))] + \frac{50}{200} [\sin(200t + 100\tau + 2\theta) - \right. \\
 &\left. \sin(200(-T) + 100\tau + 2\theta)] \right\} \\
 &= \frac{1}{2T} \left\{ [50 \cos(100\tau) (2T)] \right. \\
 &\quad \left. + \frac{1}{4} [\sin(200t + 100\tau + 2\theta) - \sin(-200T + 100\tau + 2\theta)] \right\} \\
 \lim_{T \rightarrow \infty} \overline{X_T} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left\{ [50 \cos(100\tau) (2T)] \right. \\
 &\quad \left. + \frac{1}{4} [\sin(200t + 100\tau + 2\theta) - \sin(-200T + 100\tau + 2\theta)] \right\} \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} 50 \cos(100\tau) (2T) + \lim_{T \rightarrow \infty} \frac{1}{8T} [\sin(200t + 100\tau + 2\theta) \\
 &\quad - \sin(-200T + 100\tau + 2\theta)] \\
 &= \lim_{T \rightarrow \infty} 50 \cos(100\tau) + \lim_{T \rightarrow \infty} \frac{1}{8T} [\sin(200t + 100\tau + 2\theta) \\
 &\quad - \sin(-200T + 100\tau + 2\theta)] \\
 &= 50 \cos 100\tau + 0 \\
 &= 50 \cos 100\tau
 \end{aligned}$$

$$R_{XX}(t, t + \tau) = \lim_{T \rightarrow \infty} \overline{X_T}$$

Hence  $X(t)$  is a correlation ergodic.

**2. Find the ACF of the periodic time function  $X(t) = A \sin \omega t$ .**

**Solution:**

Since periodic time function  $X(t)$  is given, we use time auto correlation function.

The ACF of the process is given by  $R_{XX}(t_1, t_2) = \lim_{T \rightarrow \infty} \overline{X_T}$

To find  $\lim_{T \rightarrow \infty} \overline{X_T}$

$$\begin{aligned}
 \overline{X_T} &= \frac{1}{2T} \int_{-T}^T X(t) X(t + \tau) dt \\
 &= \frac{1}{2T} \int_{-T}^T A \sin \omega t A \sin(\omega t + \omega \tau) dt \\
 &= \frac{A^2}{2T} \int_{-T}^T \sin \omega t \sin(\omega t + \omega \tau) dt \\
 &= \frac{A^2}{4T} \int_{-T}^T [\cos(\omega t - \omega t - \omega \tau) - \cos(\omega t + \omega t + \omega \tau)] dt \\
 &= \frac{A^2}{4T} \int_{-T}^T [\cos(-\omega \tau) - \cos(2\omega t + \omega \tau)] dt \\
 &= \frac{A^2}{4T} \left[ \cos \omega \tau (t)_{-T}^T - \left( \frac{\sin(2\omega t + \omega \tau)}{2\omega} \right)_{-T}^T \right]
 \end{aligned}$$

$$= \frac{A^2}{4T} \left[ \cos\omega\tau(2T) - \frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega} \right]$$

$$= \frac{A^2}{4T} \cos\omega\tau(2T) + \frac{A^2}{4T} \left[ -\frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega} \right]$$

The ACF of the process is given by

$$R_{XX}(t_1, t_2) = \lim_{T \rightarrow \infty} \overline{X_T}$$

$$= \frac{A^2}{2} \cos\omega\tau + \lim_{T \rightarrow \infty} \frac{A^2}{4T} \left[ -\frac{\sin(2\omega T + \omega\tau)}{2\omega} + \frac{\sin(-2\omega T + \omega\tau)}{2\omega} \right]$$

$$R_{XX}(t_1, t_2) = \frac{A^2}{2} \cos\omega\tau$$

