RESIDUES

The residue of $f(z)$ at $z = z_0$ is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series of $f(z)$ about $z = z_0$

Evaluation of Residues

(i) If $z = z_0$ is a pole of order one (simple pole) for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z \to z_0} (z - z_0)\; f(z).$$

(ii) If $z = z_0$ is a pole of order n for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{\,n-1}}{dz^{n-1}} (z - z_0)^n\; f(z)$$

Example: Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$ Here, $z = -1$ is a pole of order 2.

We know that,

$$[Res\, f(z), z = z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{\,m-1}}{dz^{m-1}} (z - z_0)^m\; f(z)$$

Here, $m = 2$

$$[Res\, f(z), z = -1] = \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} (z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \lim_{z \to -1} \frac{d}{dz} [e^{2z}] \quad = \lim_{z \to -1} 2[e^{2z}] = 2\, e^{-2}$$

Example: Find the residues at $z = 0$ of the function (i) $f(z) = e^{1/z}$ (ii) $f(z) = \frac{sinz}{z^4}$ (iii) $f(z) = zcos\frac{1}{z}$

Solution:

The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansions of $f(z)$ about $z = 0$

(i) $e^{1/z} = 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = \frac{1}{1!} = 1$by definition of residue.

(ii) $f(z) = \frac{sinz}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z^5}{5!} - \;...$

$[Res\, f(z), 0\,] =$ coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{3!} = -\frac{1}{6}$ by definition of residue.

(iii) $f(z) = z\cos\frac{1}{z} = z\left[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots\right]$

$$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$$

$[Res\, f(z), 0\,] =$ coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{2!} = -\frac{1}{2}$

Example: Find the residue of $z^2 \sin\left(\frac{1}{z}\right)$ at $z = 0$

Solution:

Let $f(z) = z^2 \sin\left(\frac{1}{z}\right) = z^2\left[\frac{\left(\frac{1}{z}\right)}{1!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right] = \frac{z}{1!} - \frac{1}{6z} + \cdots$

$[Res\, f(z), 0\,] =$ coefficient of $\frac{1}{z}$ in Laurent's expansion.

$$= -\frac{1}{6}$$

Example: Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Solution:

Here, $z = 2$ is a simple pole.

$$[Res\, f(z), z = 2] = \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)}$$

$$= \lim_{z\to 2}\frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}$$

Example: Find the residue of $\frac{1-e^{-z}}{z^3}$ at $z = 0$

Solution:

Given $f(z) = \frac{1-e^{-z}}{z^3} = \frac{1-\left[1-\frac{z}{1!}+\frac{(z)^2}{2!}-\frac{(z)^3}{3!}+\frac{(z)^4}{4!}-\ \ldots\right]}{z^3}$

$$= \frac{\left[1-\frac{z}{2!}+\frac{z^2}{3!}-\frac{z^3}{4!}+\ \ldots\right]}{z^2}$$

Here, $z = 0$ is a pole of order 2.

$$[Res\, f(z), z = 0] = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}[(z)^2 f(z)]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\left[1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \ \ldots\right]\right]$$

$$= \lim_{z\to 0}\left[\frac{-1}{2!} + \frac{2z}{3!} - \frac{3z^2}{4!} + \ \ldots\right]$$

$$= \frac{-1}{2!} = -\frac{1}{2}$$