5.3 DRAG FORCE ON FLAT PLATE DUE TO BOUNDARY LAYER

Momentum Equation for Boundary Layer by Von Karman

Von Karman suggested a method based on the momentum equation by the use of which the growth of a boundary layer along a flat plate, the wall shear stress and the drag force could be determined (when the velocity distribution in the boundary layer is known).Starting from the beginning of the plate, the method can be wed for both laminar and turbulent boundary layers.

The figure below shows a fluid flowing over a thin plate (placed at zero incidence) with a free stream velocity equal to U. Consider a small length dx of the plate at a distance x from the leading edge as shown in fig. (a). Consider unit width of plate perpendicular to the direction of flow.

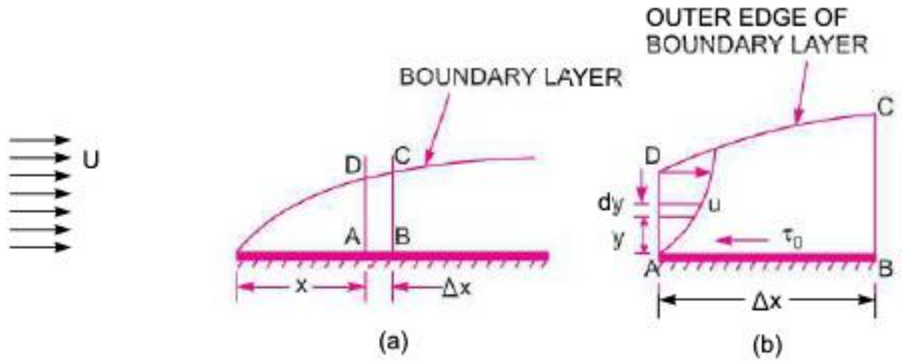

Figure 5.3.1 Drag Force on Flat Plate due to Boundary layer
[Source: "Fluid Mechanics and Hydraulics Machines" by Dr.R.K.Bansal, Page: 619]

Let ABCD be a small element of a boundary layer (the edge DC represents the outer edge of the boundary layer).
Mass rate of fluid entering through AD

$$= \int_0^\delta \rho u dy + \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx --------(ii) \left[\begin{array}{l} i.e.(mass\ through\ AD) + \frac{d}{dx} \\ (mass\ through\ AD) x\ dx \end{array}\right]$$

∴ Mass rate of fluid entering the control volume through the surface DC
= mass rate of fluid through BC – Mass rate of fluid through AD

$$= \int_0^\delta \rho u dy + \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx - \int_0^\delta \rho u dy -------(iii)$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx ------------------(iv)$$

The fluid is entering through DC with a uniform velocity U.
Momentum rate of fluid entering the control volume of X-direction through AD.

$$\int_0^\delta \rho u^2 dy -----------(v)$$

Momentum rate of fluid leaving the Control Volume in X-direction through BC

$$= \int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right]dx ----------(vi)$$

Momentum rate of fluid entering the control volume through DC in X-direction

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u dy\right]dx \times U \quad (\because Velocity = U) ----------(vii)$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u U dy\right]dx -----------(viii)$$

$\therefore$ Rate of change of momentum of Control Volume = Momentum rate of fluid through BC – Momentum rate of fluid through AD –Momentum of fluid through DC

$$= \int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right]dx - \int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u U dy\right]dx ------(ix)$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy - \int_0^{\delta} \rho u U dy \;\; dx\right] -----------$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \left(\rho u^2 dy - \rho u U dy\right)\right]dx$$

$$= \frac{d}{dx}\left[\rho \int_0^{\delta} \left(u^2 - uU\right)dy\right]dx --------------------(x)$$

As per momentum principle, the rate of change of momentum on the control volume BCD must be equal to the total force on the control volume in the same direction. The only external force acting on the control volume is the share force acting on the side AB in the direction B to A (fig. b) above). The value of this force (drag force) is given by,

$$\Delta F_D = \tau_o \times dx$$

Thus the total external force in the direction of the rate of change of momentum

$$= -\tau_o \times dx -------(xi)$$

Equating equation (x) and (xi), we have

$$-\tau_o \times dx = \rho \frac{d}{dx}\left[\int_{\delta}^{\delta} \left(u^2 - uU\right)dy\right]dx$$

or

$$\rho \frac{d}{dx}\left[\int_{\delta}^{\delta} \left(u^2 - uU\right)dy\right]$$

$$or, = \rho \frac{d}{dx}\left[\int_{\delta}^{\delta} \left(uU - u^2\right)dy\right]$$

$$= \rho \frac{d}{dx}\left[\int_0^{\delta} U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right]$$

$$= \rho U \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

$$or \quad \frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)\right]dy -------(xiii)$$

But,

$$\int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = momentum\ thickness\ \theta$$

$$\therefore \frac{\tau_o}{\rho U^2} = \frac{d\theta}{dx} --------(xvii)$$

This equation is known as von Karman momentum equation for boundary layer flow and it is used to find out the frictional drag on smooth flat plate for both laminar and turbulent boundary layer.

The following boundary conditions must be satisfied for any assumed velocity distribution.

(i) At the surface of the plate $y = 0,\ U = 0, \frac{du}{dy} = finite\ value$

(ii) At the outer edge of boundary layer $y = \delta, u = U, y = \delta, \frac{du}{dy} = 0$

The sheer stress, τ_o for a given velocity profile in laminar, transition or turbulent zone is obtained from equations (xii) and (xiii) above. Then drag force on a small distance dx of a plate is given by

$$\Delta F_D = shear\ stress \times area$$

$$= \tau_o \times (B \times dx) = \tau_o \times B \times dx\ \ [assuming\ width\ of\ plate\ as\ unity]$$

$$where,\ B = width\ of\ the\ plate$$

$$\therefore Total\ drag\ on\ the\ plate\ of\ length\ L\ one\ side,$$

$$F_D = \int \Delta F_D = \int_0^L \tau_o \times B \times dx$$

- The ratio of the shear stress to the quantity $\frac{1}{2}\ell u^2$ is known as the Local co-efficient of drag" (or co-efficient of skin fraction) and is denoted by C_D^* i.e.

$$C_D^* = \frac{\tau_o}{\frac{1}{2}\rho u^2}$$

- The ratio of the total drag force to the quantity $\frac{1}{2}\ell u^2$ is called 'Average-coefficient of drag' and is denoted by C_D i.e. $C_D^* = \frac{F_D}{\frac{1}{2}\rho A U^2}$

ℓ =Mass density of fluid

A = Area of surface/plate, and

U = free stream velocity