UNIT-IV

FOURIER TRANSFORMS

Evaluate $\int_0^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)}$ using transform methods.

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

Then $F_c\{ s \} = \sqrt{(2/\pi)} \int_0^{\infty} e^{-ax} \cos sx \, dx.$

$$= \sqrt{(2/\pi)} \frac{a}{a^2 + s^2}.$$

Similarly, $G_c\{ s \} = \sqrt{(2/\pi)} \frac{b}{b^2 + s^2}.$

Now using Parseval‟s identity for Fourier cosine transforms,

i.e., $\int_0^{\infty} F_c(s) . G_c(s) \, ds = \int_0^{\infty} f(x) \, g(x) dx.$

we have, $\frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2 + s^2)(b^2 + s^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$

or $\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(a^2 + s^2)(b^2 + s^2)} = \left. \frac{}{-(a+b)} \right|_0^{\infty}$

$$= 1 / (a+b)$$

Thus, $\int_0^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)}$

Example 14

Using Parseval‟s identity, evaluate the integrals

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} \quad \text{and} \quad \int_0^\infty \frac{x^2}{(a^2+x^2)^2}\, dx \quad \text{if } a > 0$$

Let $f(x) = e^{-ax}$

Then $F_s(s) = \sqrt{(2/\pi)}\ \dfrac{s}{a^2+s^2}$,

$F_c(s) = \sqrt{(2/\pi)}\ \dfrac{a}{a^2+s^2}$

Now, Using Parseval‟s identity for sine transforms,

$$\text{i.e.,} \quad \int_0^\infty |F_s(s)|^2\, ds = \int_0^\infty |f(x)|^2\, dx .$$

$$\text{we get,} \quad (2/\pi) \int_0^\infty \frac{s^2}{(a^2+s^2)^2}\, ds = \int_0^\infty e^{-2ax}\, dx$$

$$\text{or} \quad (2/\pi) \int_0^\infty \frac{s^2}{(a^2+s^2)^2}\, ds = \int_0^\infty = \frac{1}{2a}$$

$$\text{Thus} \quad \int_0^\infty \frac{x^2}{(a^2+x^2)^2}\, dx = \frac{\pi}{4a}, \quad \text{if } a > 0$$

Now, Using Parseval‟s identity for cosine transforms,

$$\text{i.e.,} \quad \int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |f(x)|^2\, dx .$$

$$\text{we get,} \quad (2/\pi) \int_0^\infty \frac{a^2}{(a^2+s^2)^2}\, ds = \int_0^\infty e^{-2ax}\, dx$$

$$\text{or} \quad (2a^2/\pi) \int_0^\infty \frac{ds}{(a^2+s^2)^2} = \frac{1}{2a}$$

$$\text{Thus,} \quad \int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}, \quad \text{if } a > 0$$

Exercises

1. Find the Fourier sine transform of the function

$$f(x) = \begin{cases} \sin x, & 0 \le x < a. \\ 0, & x > a \end{cases}$$

2. Find the Fourier cosine transform of e^{-x} and hence deduce by using the inversionformula

$$\int_0^\infty \frac{\cos \alpha x \, dx}{(1 + x^2)} = \frac{\pi}{2} e^{-\alpha}$$

3. Find the Fourier cosine transform of $e^{-ax}\sin ax$.

4. Find the Fourier cosine transform of $e^{-2x} + 3e^{-x}$

5. Find the Fourier cosine transform of
   (i) e^{-ax} / x (ii) $(e^{-ax} - e^{-bx}) / x$

6. Find, when $n > 0$
   (i) $F_s[x^{n-1}]$ and (ii) $F_c[x^{n-1}]$ Hint: $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$,n>0,a>0

7. Find $F_c[xe^{-ax}]$ and $F_s[xe^{-ax}]$

8. Show that the Fourier sine transform of $1 / (1 + x^2)$ is $\sqrt{(\pi/2)}\, e^{-s}$.

9. Show that the Fourier sine transform of $x / (1 + x^2)$ is $\sqrt{(\pi/2)}\, e^{-s}$.

10. Show that $x e^{-x^2/2}$ is self reciprocal with respect to Fourier sine transform.

11. Using transform methods to evaluate

(i) $$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$$ and