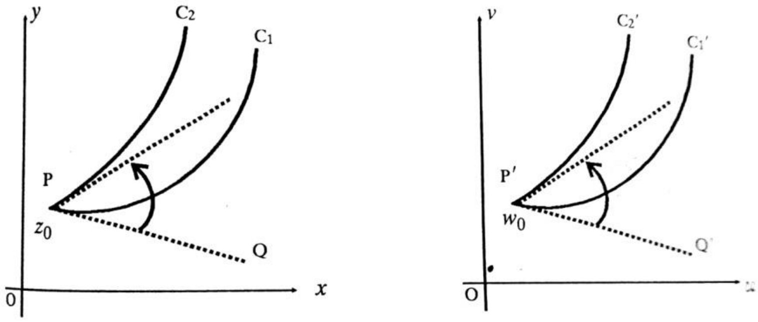
CONFORMAL MAPPING-MAPPING BY FUNCTIONS

Definition: Conformal Mapping

A transformation that preserves angels between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.



Some standard transformations

Translation:

The transformation $w = C + z$, where C is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$ and $C = a + ib$

Given $w = z + C$,

$(i.e.)\ u + iv = x + iy + a + ib$

$\Rightarrow u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get $u = x + a, v = y + b$

Hence the image of any point $p(x, y)$in the z−plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z−plane is mapped onto the w plane.

If we assume that the w−plane is super imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|C| = \sqrt{a^2 + b^2}$ in the direction of C i.e., translated by the vector representing C.

Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.

Example: What is the region of the w plane into which the rectangular region in the Z plane bounded by the lines $x = 0, y = 0,\ x = 1\ and\ y = 2$ is mapped under the transformation $w = z + (2 - i)$

Solution:

Given $w = z + (2 - i)$

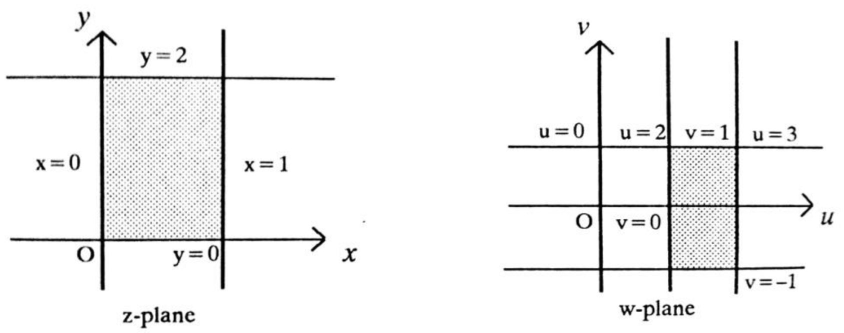
$(i.e.)\ u + iv = x + iy + (2 - i) = (x + 2) + i(y - 1)$

Equating the real and imaginary parts

$$u = x + 2,\ v = y - 1$$

Given boundary lines are	transformed boundary lines are
$x = 0$	$u = 0 + 2 = 2$
$y = 0$	$v = 0 - 1 = -1$
$x = 1$	$u = 1 + 2 = 3$
$y = 2$	$v = 2 - 1 = 1$

Hence, the lines $x = 0, y = 0, x = 1,$ and $y = 2$ are mapped into the lines $u = 2, v = -1,$ $u = 3, and\ v = 1$ respectively which form a rectangle in the w plane.



Example: Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution:

Given $w = z + 2 + 4i$

$(i.e.)\ u + iv = x + iy + 2 + 4i$

$= (x + 2) + i(y + 4)$

Equating the real and imaginary parts, we get

$u = x + 2, v = y + 4,$

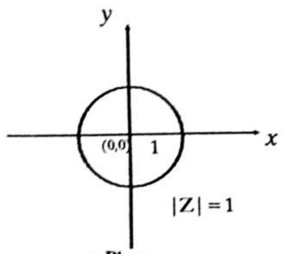
$x = u - 2, y = v - 4,$

Given $|z| = 1$

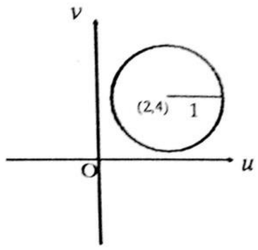
$(i.e.)\ x^2 + y^2 = 1$

$(u - 2)^2 + (v - 4)^2 = 1$

Hence, the circle $x^2 + y^2 = 1$is mapped into $(u - 2)^2 + (v - 4)^2 = 1$ in w plane which is also a circle with centre (2, 4)and radius 1.



z-Plane



w-Plane

2. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

This means that the magnitude of the vector representing z is magnified by $a = |c|$ and its direction is rotated through angle $\alpha = amp\ (c)$. Hence the transformation consists of a magnification and a rotation.

Example: Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ is transformed under the transformation $w = 2z$.

Solution:

Let $w = u + iv$

$z = x + iy$

Given $w = 2z$

$u + iv = 2(x + iy)$

$u + iv = 2x + i2y$

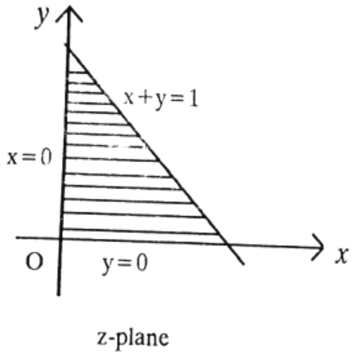
$u = 2x \Rightarrow x = \frac{u}{2}, v = 2y \Rightarrow y = \frac{v}{2}$

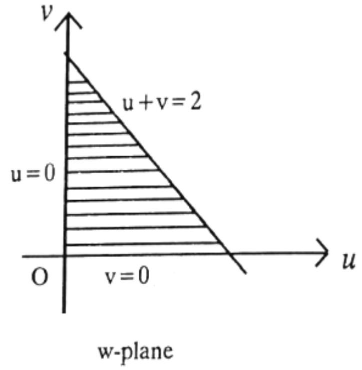
Given region (D) whose boundary lines are		Transformed region D' whose boundary lines are
$x = 0$	$\Rightarrow$	$u = 0$
$y = 0$	$\Rightarrow$	$v = 0$
$x + y = 1$	$\Rightarrow$	$\frac{u}{2} + \frac{v}{2} = 1 [\because x = \frac{u}{2}, y = \frac{v}{2}]$ $(i.e.)\ u + v = 2$

In the z plane the line $x = 0$ is transformed into $u = 0$in the w plane.

In the z plane the line $y = 0$ is transformed into $v = 0$in the w plane.

In the z plane the line $x + y =$ is transformed into$u + v = 2$ in the w plane.





Example: Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Solution:

Given $w = 5z$

$|w| = 5|z|$

i.e., $|w| = 5\lambda \quad [\because |z| = \lambda]$

Hence, the image of $|z| = \lambda$ in the z plane is transformed into $|w| = 5\lambda$ in the w plane under the transformation $w = 5z$.

Example: Find the image of the circle $|z| = 3$ under the transformation $w = 2z$

Solution:

Given $w = 2z, \ |z| = 3$

$|w| = (2)|z|$

$= (2)(3), \quad$ Since $|z| = 3$

$= 6$

Hence, the image of $|z| = 3$ in the z plane is transformed into $|w| = 6$ w plane under the transformation $w = 2z$.

Example: Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$.

Solution:

Given $w = (1 - i)z$.

$u + v = (1 - i)(x + iy)$

$= x + iy - ix + y$

$= (x + y) + i(y - x)$

i.e., $u = x + y, \quad v = y - x$

$u + v = 2y \qquad u - v = 2x$

$$y = \frac{u+v}{2} \qquad x = \frac{u-v}{2}$$

Hence, image region $y > 1$ is $\frac{u+v}{2} > 1$ i.e., $u + v > 2$ in the w plane.

3. Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion w.r.to the unit circle $|z| = 1$, followed by reflection in the real axis.

$$\Rightarrow w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$\Rightarrow x + iy = \frac{1}{u^2+v^2}$$

$$\Rightarrow x = \frac{1}{u^2+v^2} \qquad \ldots (1)$$

$$\Rightarrow y = \frac{-v}{u^2+v^2} \qquad \ldots (2)$$

We know that, the general equation of circle in z plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (3)$$

Substitute, (1) and (2) in (3)we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad \ldots (4)$$

which is the equation of the circle in w plane

Hence, under the transformation $w = \frac{1}{z}$ a circle in z plane transforms to another circle in the w plane. When the circle passes through the origin we have $c = 0$ in (3). When $c = 0$, equation (4) gives a straight line.

Example: Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$

Solution:

Given $|z - 2i| = 2$ .....(1) is a circle.

Centre = (0,2)

radius = 2

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w} - 2i\right| = 2$

$\Rightarrow |1 - 2wi| = 2|w|$

$\Rightarrow |1 - 2(u + iv)i| = 2|u + iv|$

$$\Rightarrow |1 - 2ui + 2v| = 2|u + iv|$$

$$\Rightarrow |1 + 2v - 2ui| = 2|u + iv|$$

$$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

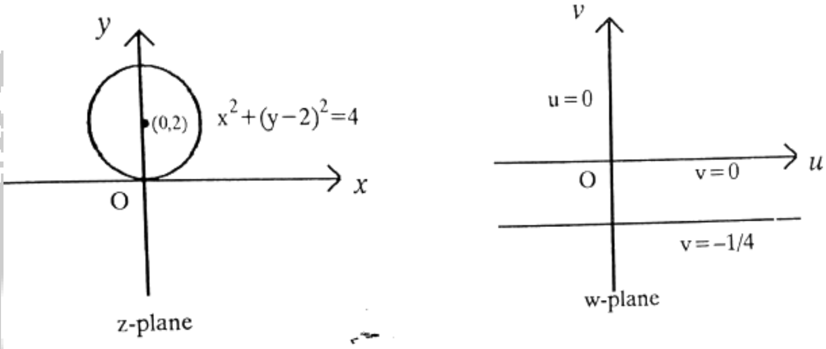
$$\Rightarrow (1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = -\frac{1}{4}$$

Which is a straight line in w plane.



Example: Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$

Solution:

Given $|z - 1| = 1$ .....(1) is a circle.

Centre =(1,0)

radius = 1

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w} - 1\right| = 1$

$$\Rightarrow |1 - w| = |w|$$

$$\Rightarrow |1 - (u + iv)| = |u + iv|$$

$$\Rightarrow |1 - u + iv| = |u + iv|$$

$$\Rightarrow \sqrt{(1 - u)^2 + (-v)^2} = \sqrt{u^2 + v^2}$$

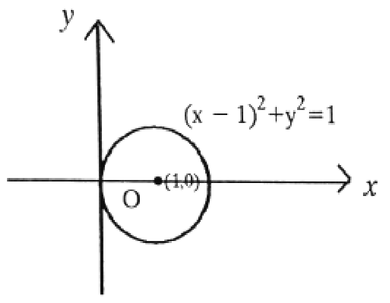
$$\Rightarrow (1 - u)^2 + v^2 = u^2 + v^2$$

$$\Rightarrow 1 + u^2 - 2v + v^2 = u^2 + v^2$$

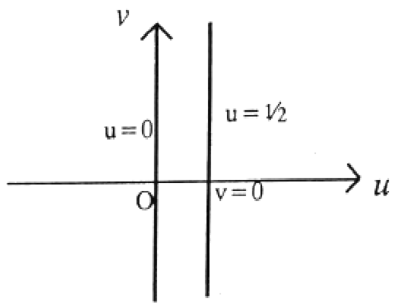
$$\Rightarrow 2u = 1$$

$$\Rightarrow u = \frac{1}{2}$$

which is a straight line in the w- plane



z-plane



w-plane

Example: Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ **(ii)** $0 < y < \frac{1}{2}$ **under the transformation** $w = \frac{1}{z}$

Solution :

Given $w = \frac{1}{z}$ (given)

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$x = \frac{u}{u^2+v^2} \ldots.(1), y = \frac{-v}{u^2+v^2} \ldots.(2)$$

(i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$\frac{1}{4} = \frac{-v}{u^2+v^2}$ by (2)

$\Rightarrow u^2 + v^2 = -4v$

$\Rightarrow u^2 + v^2 + 4v = 0$

$\Rightarrow u^2 + (v+2)^2 = 4$

which is a circle whose centre is at $(0, -2)$ in the w plane and radius is 2k.

when $y = \frac{1}{2}$

$\frac{1}{2} = \frac{-v}{u^2+v^2}$ by (2)

$\Rightarrow u^2 + v^2 = -2v$

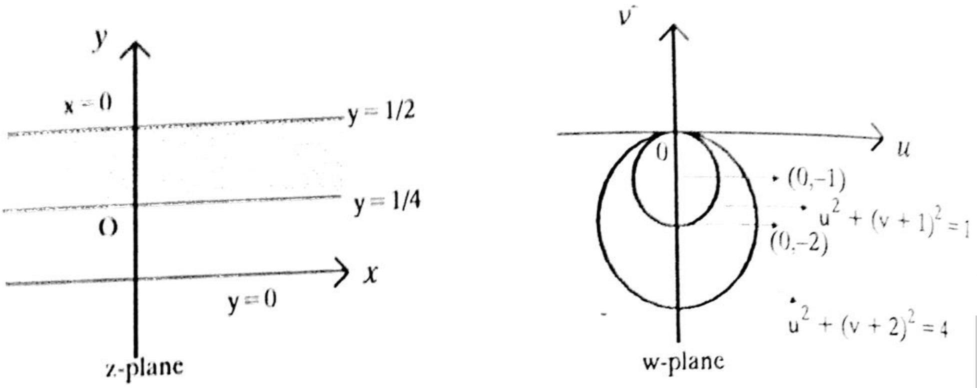
$\Rightarrow u^2 + v^2 + 2v = 0$

$\Rightarrow u^2 + (v+1)^2 = 0$

$$\Rightarrow u^2 + (v+1)^2 = 1 \qquad \ldots\ldots(3)$$

which is a circle whose centre is at $(0,-1)$ in the w plane and unit radius

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w plane.



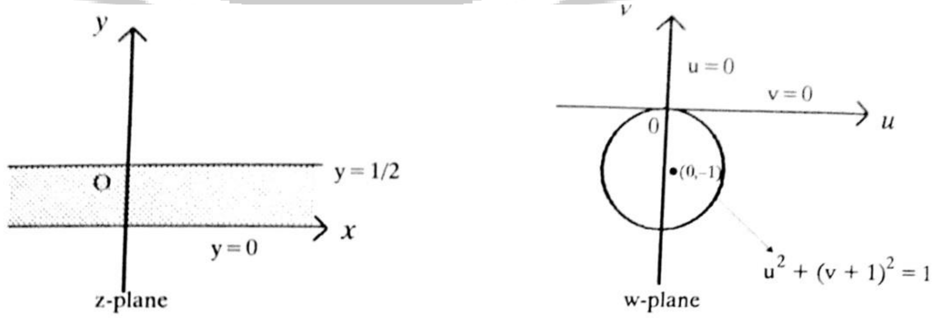
ii) Given strip is $0 < y < \frac{1}{2}$

when $y = 0$

$\Rightarrow v = 0$ by (2)

when $y = \frac{1}{2}$ we get $u^2 + (v+1)^2 = 1$ by (3)

Hence, the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w plane.



Example: Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$.

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \ldots.(1), y = \frac{-v}{u^2+v^2} \quad \ldots(2)$$

Given $x = 2$ in the z plane.

$$\therefore 2 = \frac{u}{u^2+v^2} \quad \text{by (1)}$$

$$2(u^2 + v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$

$\therefore x = 2$ in the z plane is transformed into a circle in the w plane.

Example: What will be the image of a circle containing the origin(i.e., circle passing through the origin) in the XY plane under the transformation $w = \frac{1}{z}$?

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \quad \ldots(1),$$

$$y = \frac{-v}{u^2+v^2} \quad \ldots(2)$$

Given region is circle $x^2 + y^2 = a^2$ in z plane.

Substitute, (1) and (2), we get

$$\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\frac{1}{(u^2+v^2)} = a^2$$

$$u^2 + v^2 = \frac{1}{a^2}$$

Therefore the image of circle passing through the origin in the XY −plane is a circle passing through the origin in the w − plane.

Example: Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \quad \ldots.(1), \qquad y = \frac{-v}{u^2+v^2} \quad \ldots.(2)$$

Given $1 < x < 2$

When $x = 1$

$$\Rightarrow 1 = \frac{u}{u^2+v^2} \quad \text{by } \ldots.(1)$$

$$\Rightarrow u^2 + v^2 = u$$

$$\Rightarrow u^2 + v^2 - u = 0$$

which is a circle whose centre is $\left(\frac{1}{2}, 0\right)$ and is $\frac{1}{2}$

When $x = 2$

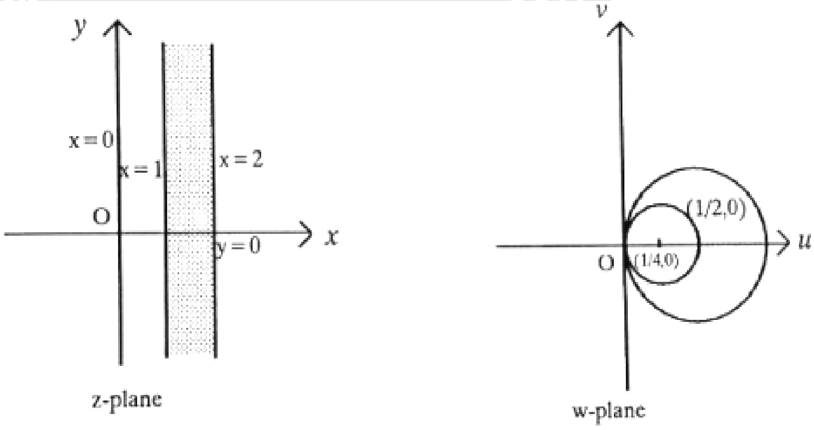
$$\Rightarrow 2 = \frac{u}{u^2+v^2} \quad \text{by } \ldots.(1)$$

$$\Rightarrow u^2 + v^2 = \frac{u}{2}$$

$$\Rightarrow u^2 + v^2 - \frac{u}{2} = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and is $\frac{1}{4}$

Hence, the infinite strip $1 < x < 2$ is transformed into the region in between the circles in the $w-$plane.



4. Transformation $w = z^2$

Problems based on $w = z^2$

Example: Discuss the transformation $w = z^2$.

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2 = x^2 + (iy)^2 + i2xy = x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad \ldots.(1), \quad v = 2xy \quad \ldots.(2)$$

Elimination:

$$(2) \Rightarrow x = \frac{v}{2y}$$

$$(1) \Rightarrow u = \left(\frac{v}{2y}\right)^2 - y^2$$

$$\Rightarrow u = \frac{v^2}{4y^2} - y^2$$

$$\Rightarrow 4uy^2 = v^2 - 4y^4$$

$$\Rightarrow 4uy^2 + 4y^4 = v^2$$

$$\Rightarrow y^2[4u + 4y^2] = v^2$$

$$\Rightarrow 4y^2[u + y^2] = v^2$$

$$\Rightarrow v^2 = 4y^2(y^2 + u)$$

when $y = c \ (\neq 0)$, we get

$$v^2 = 4c^2(u + c^2)$$

which is a parabola whose vertex at $(-c^2, 0)$ and focus at (0,0)

Hence, the lines parallel to X-axis in the z plane is mapped into family of confocal parabolas in the w plane.

when $y = 0$, we get $v^2 = 0$ i.e., $v = 0, u = x^2$ i.e., $u > 0$

Hence, the line $y = 0$, in the z plane are mapped into $v = 0$, in the w plane.

Elimination:

$$(2) \Rightarrow y = \frac{v}{2x}$$

$$(1) \Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow u = x^2 - \frac{v^2}{4x^2}$$

$$\Rightarrow \frac{v^2}{4x^2} = x^2 - u$$

$$\Rightarrow v^2 = (4x^2)(x^2 - u)$$

when $x = c\ (\neq 0)$, we get $v^2 = 4c^2(c^2 - u) = -4c^2(u - c^2)$

which is a parabola whose vertex at $(c^2, 0)$ and focus at (0,0) and axis lies along the u −axis and which is open to the left.

Hence, the lines parallel to y axis in the z plane are mapped into confocal parabolas in the w plane when $x = 0$, we get $v^2 = 0$. i.e., $v = 0, u = -y^2$ i.e., $u < 0$

i.e., the map of the entire y axis in the negative part or the left half of the u −axis.

Example: Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation $w = z^2$ if

$\boldsymbol{w = u + iv}$

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad \ldots\ldots (1)$$

$$v = 2xy \quad \ldots\ldots (2)$$

Given $x^2 - y^2 = 10$

i.e., $u = 10$

Hence, the image of the hyperbola $x^2 - y^2 = 10$ in the z plane is mapped into $u = 10$ in the w plane which is a straight line.

Example: Find the critical points of the transformation $w^2 = (z - \alpha)\,(z - \beta)$.

Solution:

Given $w^2 = (z - \alpha)\,(z - \beta)$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow 2w\frac{dw}{dz} = (z - \alpha) + (z - \beta)$$

$$= 2z - (\alpha + \beta)$$

$$\Rightarrow \frac{dw}{dz} = \frac{2z - (\alpha + \beta)}{2w} \quad \ldots (2)$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z - (\alpha + \beta)}{2w} = 0$$

$$\Rightarrow 2z - (\alpha + \beta) = 0$$

$$\Rightarrow 2z = \alpha + \beta$$

$$\Rightarrow \ z = \frac{\alpha+\beta}{2}$$

Case $(ii) \frac{dz}{dw} = 0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \frac{w}{z-\frac{\alpha+\beta}{2}} = 0$$

$$\Rightarrow w = 0 \ \Rightarrow (z-\alpha)\,(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

∴ The critical points are $\frac{\alpha+\beta}{2}, \alpha$ and β.

Example: Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$.

Solution:

Given $w = z^2 + \frac{1}{z^2}$ ... (1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \ \frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4-2}{z^3}$$

Case $(i) \frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow z = \pm 1, \pm i$$

Case $(ii) \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{2z^4-2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$$

∴ The critical points are ± 1 , $\pm i$, 0

Example: Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation.

Solution:

Given $|z| = 1$ is a circle

Centre = $(0,0)$

Radius = 1

Given $w = \frac{z}{1-z}$

$$\Rightarrow z = \frac{w}{w+1}$$

$$\Rightarrow |z| = \left|\frac{w}{w+1}\right| = \frac{|w|}{|w+1|}$$

Given $|z| = 1$

$$\Rightarrow \frac{|w|}{|w+1|} = 1$$

$$\Rightarrow |w| = |w+1|$$

$$\Rightarrow |u+iv| = |u+iv+1|$$

$$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$\Rightarrow u^2+v^2 = (u+1)^2+v^2$$

$$\Rightarrow u^2+v^2 = u^2+2u+1+v^2$$

$$\Rightarrow 0 = 2u+1$$

$$\Rightarrow u = \frac{-1}{2}$$

Further the region $|z| < 1$ transforms into $u > \frac{-1}{2}$