4.1 Semigroups and Monoids

Define Algebraic System:

- A non – empty set G together with one or more n – ary operations say $*$ (binary) is called an Algebraic System or Algebraic Structure or Algebra.
- We denoted it by $[G, *]$.
- Note: $+, -, \cdot, \times, *, \cup, \cap$ etc are some of binary operations.

Properties of Binary Operations

Let the binary operation be $*$: $G \times G \rightarrow G$.

Then we have the following properties:

Closure Property:

a $*$ b = $x \epsilon$ G, for all $a, b\ \varepsilon\ G$.

Commutativity Property:

$a * b = b * a$, for all $a, b\ \varepsilon\ G$.

Associativity:

$(a * b) * c = a * (b * c)$, for all $a, b, c\ \varepsilon\ G$.

Identity Element:

$a * e = e * a = a$, for all $a\ \ \varepsilon\ G$.

'e' is called the identity element.

Inverse Element:

If $a * b = b * a = e$ (identity), then b is called the inverse of a and it is denoted by b = a^{-1}.

Left Cancellation law:

$a * b = a * c \Rightarrow b = c$

Right Cancellation law:

$b * a = c * a \Rightarrow b = c$

If the binary operation defined on G is + and X, then we have the following table.

For all a, b, c ε G	(G, +)	(G,×)
Commutativity	a + b = b + a	a× b = b × a
Associativity	(a + b) + c = a + (b + c)	(a× b)× c =a × (b × c)
Identity element	a + 0 = 0 + a = a (0 → identity)	a× 1 = 1 × a = a (1 →identity)
Inverse element	a + (-a) = 0 (-a→ additive inverse)	a× $\frac{1}{a} = \frac{1}{a}$ × a = 1 ($\frac{1}{a}$ → multiplicative inverse)

NOTATIONS:

- Z - the set of all integers.
- Q - the set of all rational numbers.
- R - the set of all real numbers.
- C - the set of all complex numbers.
- R^+ - the set of all positive real numbers.
- Q^+ - the set of all positive rational numbers.

Semigroups and Monoids:

Define semigroup

If a non – empty set S together with the binary operation $*$ satisfying the following properties

Closure Property:

$a * b = b * a$, for all $a, b \varepsilon S$.

Associativity:

$(a * b) * c = a * (b * c)$, for all $a, b, c \varepsilon S$.

Then $(S,*)$ is called a semigroup.

Monoid:

A semigroup $(S,*)$ with an identity element with respect to $*$ is called Monoid. It is denoted by $(M,*)$.

In other words, a non – empty set 'M' with respect to $*$ is said to be a monoid, if $*$ satisfies the following properties

For $a, b \in M$

Closure Property:

$a * b = b * a$, for all a, b ε M.

Associativity:

$(a * b) * c = a * (b * c)$, for all a, b, c ε M.

Identity Element:

$a * e = e * a = a$, for all a ε M.

'e' is called the identity element.