3.4 Poisson Process

Let $X(t)$ denotes the number of occurrences of a certain event in the interval $(0, t)$. Then the discrete random process $\{X(t)\}$ is called the Poisson process if it satisfies the following postulates.

Postulates of Poisson process

(i) $P[1\ occurrence\ in\ (t, t + \Delta t)] = \lambda\Delta t + O(\Delta t) = P_1(\Delta t)$

(ii) $P[0\ occurrence\ in\ (t, t + \Delta t)] = 1 - \lambda\Delta t + O(\Delta t) = P_0(\Delta t)$

(iii) $P[2\ or\ more\ occurrences\ in\ (t, t + \Delta t)] = O(t)$

Some Applications of Poisson Process:

(i) Arrival of customers in simple queuing system

(ii) The number of wrong telephone calls at a switch board.

(iii) The number of passengers entering a railway station on a given day.

(iv) The emission of radioactive particles.

Probability law of Poisson process:

Let λ be the number of customers per unit time. $P_m(t)$ be the probability of m occurences, of the event in the interval (0, t).

$P_0(t)$ denotes the probability of 0 occurences, in the interval (0, t).

Probability of zero occurrence of the event in the time interval 0 to $t + \Delta t$ is given by $P_0(t + \Delta t)$.

Also $P_0(t+\Delta t)$ can be rewritten as

$P_0(t+\Delta t)$ = Probability of zero occurrence in the interval $(0,t)$ and also in the interval $(t, t+\Delta t\ )$

i.e., $P_0(t+\Delta t) = P_0(1-\lambda\Delta t)$

$$= P_0(t) - \lambda \Delta t P_0(t)$$

$$\Rightarrow P_0(t+\Delta t) - P_0(t) = -\lambda \Delta t P_0(t)$$

$$\Rightarrow \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

Taking lim as $\Delta t \to 0$

$$\Rightarrow \lim_{n\to 0} \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\Rightarrow \frac{d}{dt}[P_0(t)] = -\lambda P_0(t)$$

$$\Rightarrow \frac{d[P_0(t)]}{P_0(t)} = -\lambda dt$$

Integrating on both sides

$$\Rightarrow \int \frac{d[P_0(t)]}{P_0(t)} = \int -\lambda dt$$

$$\Rightarrow \log P_0(t) = -\lambda t + c$$

$\Rightarrow P_0(t) = e^{-\lambda t} A$ where $A = e^c$ ... (1)

To find A, Put $t = \ 0$

$\Rightarrow P_0(0) = A$

$\Rightarrow A = 1$

$(1) \Rightarrow P_0(0) = e^{-\lambda t}$

Now $P_m(t) = P[X(t) = m]$

= P(m occurrences of the event in $(0, t)$)

$P_m(t + \Delta t) = P[X(t + \Delta t) = m]$

= P(m occurrences of the event in $(0, t + \Delta t)$)

= P(m – 1 occurrences in $(0, t)$ and 1 occurrence in $(t, t + \Delta t)$)+ P(m occurrences of the event in $(0, t)$ and 0 occurrence in $(t, t + \Delta t)$)

$= P_{m-1}(t)\lambda\Delta t + P_m(t)(1 - \lambda\Delta t)$

$= P_{m-1}(t)\lambda\Delta t + P_m(t) - \lambda P_m(t)\Delta t$

$\Rightarrow P_m(t + \Delta t) - P_m(t) = \lambda\Delta t[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$

Taking lim as $\Delta t \to 0$

$\lim_{\Delta t \to 0} \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{d}{dt}[P_m(t)] = \lambda[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{d}{dt} P_m(t) + \lambda P_m(t) = \lambda P_{m-1}(t)$

$\Rightarrow P_m(t)e^{\lambda t} = \int \lambda P_{m-1}(t)\, e^{\lambda t} dt$

Put $m = 1$

$\Rightarrow P_1(t)e^{\lambda t} = \int \lambda P_0(t)\, e^{\lambda t} dt$

$= \int \lambda e^{-\lambda t}\, e^{\lambda t} dt$

$= \int \lambda\, dt$

$\Rightarrow P_1(t)e^{\lambda t} = \lambda dt$

$\Rightarrow P_1(t)e^{\lambda t} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!}$

This is the first order Poisson probability distribution.

In general $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

Properties of Poisson Process:

Property:1

Poisson Process is a Markov process.

Proof:

Let us take the conditional probability distribution of $X(t_3)$ given the past values of $X(t_2)$ and $X(t_1)$.

Assume that $t_3 > t_2 > t_1$ and $n_3 > n_2 > n_1$

Consider $P[X(t_3) = n_3/X(t_2) = n_2, X(t_1) = n_1\ ]$

$$= \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda(t_3-t_2)}.\lambda^{n_3-n_2}\cdot(t_3-t_2)^{n_3-n_2}}{(n_2-n_3)!}$$

$$= P[X(t_3) = n_3/X(t_2) = n_2 \,]$$

Hence the conditional probability distribution $X(t_3)$ given that values of the process $X(t_2)$ and $X(t_1)$ depends only on the most recent value $X(t_2)$ of the process.

Hence Poisson process is a Markov process.

Hence the proof.

Property: 2

Additive Property

The sum of two independent Poisson processes is a Poisson Process.

Proof:

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson process with parameter $\lambda_1 t$ and $\lambda_2 t$ respectively.

Let $X(t) = X_1(t) + X_2(t)$

Now $P[X(t) = n] = P\big[\big(X_1(t) + X_2(t)\big) = n\big]$

$\Rightarrow \sum_{r=0}^{n} P[X_1(t) = r]P[X_2(t) = n - r]$

$\Rightarrow \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (\lambda_1 t)^r (\lambda_2 t)^{n-r}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n$

Which is a Poisson process with parameter $\lambda_1 + \lambda_2$.

Hence the proof.

Property: 3

Difference of two independent Poisson processes not a Poisson Process.

Proof:

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson process with parameter $\lambda_1 t$ and $\lambda_2 t$ respectively.

Let $X(t) = X_1(t) + X_2(t)$

Now $E[X(t)] = E[X_1(t)] - E[X_2(t)]$

$= \lambda_1 t - \lambda_2 t$

Now, $E[X^2(t)] = E\left[\left(X_1(t) - X_2(t)\right)^2\right]$

$$= E\left[{X_1}^2 t + {X_2}^2 t - 2X_1(t)X_2(t)\right]$$

$$= E\left[{X_1}^2 t\right] + E\left[{X_2}^2 t\right] - 2E[X_1(t)X_2(t)]$$

$$= {\lambda_1}^2 t^2 + \lambda_1 t + {\lambda_2}^2 t^2 + \lambda_2 t - 2E[X_1(t)]E[X_2(t)]$$

$$= {\lambda_1}^2 t^2 + {\lambda_2}^2 t^2 + \lambda_1 t + +\lambda_2 t - 2\lambda_1 t \lambda_2 t$$

$$= (\lambda_1 + \lambda_2)t + \left({\lambda_1}^2 + {\lambda_2}^2\right)t^2 - 2\lambda_1\lambda_2 t^2$$

$$= (\lambda_1 + \lambda_2)t + \left({\lambda_1}^2 + {\lambda_2}^2 - 2\lambda_1\lambda_2\right)t^2$$

$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

$$\neq (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Hence $X(t) = X_1(t) + X_2(t)$ is not a Poisson process.

Hence the proof.

Property: 4

Distribution of the interval arrival time

The inter arrival time of a Poisson process with parameter λ follows exponential distribution with mean $\frac{1}{\lambda}$.

Proof:

Let the two consecutive events be E_i and E_{i+1}.

Let E_i take place at time t_i and E_{i+1} take place at time $t_i + T$.

Let T be the interval between the occurrences E_i and E_{i+1} then T is a continuous random variable.

Now, $P(T > t) = P(t < T)$

$= P(E_{i+1}$ does not occur in $(t_i, t_i + T\ ))$

= P(no event occurs in the interval of length t)

$= P[X(t) = 0]$

$= \frac{e^{-\lambda t}(\lambda t)^0}{0!}$

$= e^{-\lambda t}$

The cumulative distribution of T is

$F(t) = P(T \leq t) = 1 - P(T > t)$

$= 1 - e^{-\lambda t}$

Hence the probability density function is

$$f(t) = \frac{d}{dt}[F(t)]$$

$= -e^{-\lambda t}(-\lambda)$

$$= \lambda e^{-\lambda t}$$

Which is the probability density function of exponential distribution with mean $\frac{1}{\lambda}$.

Hence the proof.

Property: 5

If the number of occurrences of an event E in an interval of length t is a Poisson process $X(t)$ with parameter λt and if such occurrences of E has a constant probability of being recorded and the recording are independent of each other. Then the number $N(t)$ of the recorded occurrences in time t is also a Poisson process.

Proof:

$$P[N(t) = n] = \sum_{r=0}^{n} [E\ occurs\ (n+r) times\ and\ n\ of\ them\ are\ recorded]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda t}(\lambda t)^r}{(n+r)!} (n+r)C_n p^n q^{n+r-n}$$

$$= e^{-\lambda t} \sum_{r=0}^{n} \frac{(\lambda t)^n (\lambda t)^r}{(n+r)!} \frac{(n+r)!}{n!(n+r-n)!} p^n q^r$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!} p^n \sum_{r=0}^{n} \frac{(\lambda t q)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda tp)^n}{n!}\left[1 + \frac{\lambda tp}{1!} + \frac{(\lambda tp)^2}{2!} + \dots\right]$$

$$= \frac{e^{-\lambda t}(\lambda tp)^n}{n!}e^{\lambda tq}$$

$$= \frac{e^{-\lambda t(1-q)}(\lambda tp)^n}{n!}$$

$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!}$$

= Probability density function of Poisson process

Hence $N(t)$ is a Poisson process.

Hence the proof.

Problems under Probability law of Poisson Process:

1. Suppose the customer arrive at a bank according to a Poisson Process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes. (i) Exactly 4 customer arrive. (ii)Greater than 4 customer arrive. (iii)Fewer than 4 customers arrive.

Solution:

Let $X(t)$ denotes the number of customers arrived during the interval $(0, t)$.

Then $\{X(t)\}$ follows Poisson process.

Given : $\lambda = 3/\min$ $and\ t = 2\ min.$

$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3 \dots \dots ..$

$i.e).,P(X(2) = n) = \frac{e^{-6}(6)^n}{n!}, n = 0,1,2,3 \dots ..$

(i)P(Exactly 4 customer arrive during a time interval of two minutes)

$P(X(2) = 4) = \frac{e^{-6}(6)^4}{4!} = \frac{3.2124}{24} = 0.13385$

$P(X(2) = 4) = 0.13385$

(ii)P(greater than 4 customer arrive during a time interval of two minutes) =

$P(X(2) > 4) = 1 - P(X(2) \leq 4)$

$= 1 - \{P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) +$

$P(X(2) = 3) + P(X(2) = 4)\}$

$$= 1 - \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!}\right]$$

$$= 1 - e^{-6}\left[1 + 6 + \frac{36}{2} + 36 + 54\right]$$

$$= 1 - e^{-6}[115] = 1 - 0.285 = 0.715$$

$P(X(2) > 4) = 0.715$

(iii)P(less than 4 customer arrive during a time interval of two minutes)

$P(X(2) < 4) = P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) + P(X(2) = 3)$

$$= \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!}\right]$$

$$= e^{-6}[1 + 6 + 8 + 36]$$

$$= e^{-6}[61] = 0.1512$$

$P(X(2) < 4) = 0.1512$

2. A hard disk fails in a computer system it follows a Poisson distribution with mean rate of 1 per week. Find the probability, that 2 weeks have elapsed since last failure. If we have 5 extra hard disks and the next supply is not due in 10 weeks. Find the probability that the machine will not be out of order in the next 10 weeks.

Solution:

Given $\lambda = 1$

$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3 \ldots$

To find the probability that 2 weeks have elapsed since last failure.

$P(X(2) = 0) = \frac{e^{-1\times 2}(1\times 2)^0}{0!}$

$$= e^{-2}$$

$$= 0.1353$$

Number of extra hard disk = 5

The probability that the machine will not be the probability that the machine will not be out of order in the next weeks is $P(X(10) \leq 5)$

$$= \{P(X(10) = 0) + P(X(10) = 1) + P(X(10) = 2) + P(X(10) = 3) + P(X(10) = 4) + P(X(10) = 5)\}$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!}$$

$$= e^{-10}\left[1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!} + \frac{10^5}{5!}\right]$$

$$= 0.067$$

3. A fisher man catches fish independently at a Poisson rate of 2 from a large lake with a lot fish. If he starts at 10.00am, what is the hour pribability that he catches 1 fish by 10.30am and 3 fishes by noon?

Solution:

Let $X(t)$ denotes the number of fishes caught by the fisherman in $(0, t)$. Then $\{X(t)\}$ follows a Poisson process. Given $\lambda = 2$ per hr.

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0,1,2 \ldots \ldots \ldots \infty$$

$= \frac{e^{-2t}(2t)^n}{n!}; n = 0,1,2 \dots \dots \dots \infty \ \lambda = 2$

Given a fisherman starts catching at 10.00am and fisherman catches fish independently.

$\therefore P[$ he catches one fish in 10.30am and 3 fishes in noon$]$

$= P[$ he catches one fish in 10.30am$]P[$ he catches 3 fishes in noon$]$

$= P$ [he catches one fish in 30mins$]P[$ he catches 3 fishes at 12pm$]$

$= P$ [he catches one fish in 30mins$]P[$ he catches 3 fishes in 2 hours $]$

$= P[$ he catches one fish by $\frac{1}{2}$ an hour $]P[$ he catches 3 fishes in 2 hours$]$

$[\because t$ is in hours$]$

$$= P\left[X\left(\frac{1}{2}\right) = 1\right] P[X(2) = 3] = \frac{e^{-2\frac{1}{2}}\left(2\left(\frac{1}{2}\right)\right)^1}{1!} \frac{e^{-4}4^3}{3!} = e^{-1}\frac{32e^{-4}}{3}$$

$P[$ he catches 3 fishes in noon $] = 0.07188$

4. On the average, a submarine on patrol sights 6 enemy ship per hour. Assume that the number of ships sighted in a given length of time is a Poisson variate, find the probability of sighting (1) 6 ships in the next half an hour (2) 4 ships in the next 2 hours (3) at least one ship in the next 15 minutes.

Solution:

Let $X(t)$ denote the number of submarine on patrol sights in the interval $(0, t)$.

Then $\{X(t)\}$ follows Poisson process.

Given $\lambda = 6$ per hour

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0,1,2, \dots., \infty$$

$$= \frac{e^{-6t}(6t)^n}{n!}; n = 0,1,2, \dots, \infty$$

(i) $P[$ sighting 6 ships next half an hour $] = P\left[X\left(\frac{1}{2}\right) = 6\right]$

$$= \frac{e^{-6\left(\frac{1}{2}\right)}\left(6\left(\frac{1}{2}\right)\right)^6}{6!} = \frac{e^{-3}3^6}{6!} = 0.0504$$

(ii) $P[$ sighting 4 ships in the next 2 hours $] = P[X(2) = 4]$

$$= \frac{e^{-6(2)}(6(2))^4}{n!} = \frac{e^{-12}12^4}{4!} = 0.0053$$

(iii) $P[$ sighting at least one ship in the next 15mins $] = P[$ at least one ship in the next $\frac{1}{4}hr]$

$$= P\left[X\left(\frac{1}{4}\right) \geq 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) < 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) = 0\right]$$

$$= 1 - \frac{e^{-6\left(\frac{1}{4}\right)}\left(6\left(\frac{1}{4}\right)\right)^0}{0!}$$

$$= 1 - e^{-\frac{3}{2}} = 0.975$$

Problems under $P(N(t) = n) = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$

1. A radioactive source emits particles at rate of five per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in a 4 minutes period.

Solution:

Given: $\lambda = 5/\min \ and \ t = 4, n = 10$

Probability of recording : $p = 0.6$

Let $N(t)$ be the number of particles recorded during the interval $(0, t)$

Then $\{N(t)\}$ follows a Poisson process with parameter λp.

$$P(N(t) = n) = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$$

$$= \frac{e^{-3t}(3t)^n}{n!} ; n = 0,1,2,3, ...$$

P(10 particles are recorded in a 4 minute period) = $P(N(4) = 10) = \frac{e^{-12}(12)^{10}}{10!}$

$P(N(4) = 10) = 0.1048$

2.If customer arrives at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two

consecutive arrivals is (i) more than 1 minute (ii) between 1 minute and 2 minutes and (iii)4 minutes or less.

Solution:

Given: $\lambda = 2/\min$

Let T be the interval between two consecutive arrivals.

The exponential distribution is $f(t) = 2e^{-2t}, t > 0$

(i)$P[T > 1] = \int_1^{\infty} f(t)\, dt$

$P[T > 1] = \int_1^{\infty} 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^{\infty} = -[e^{-\infty} - e^{-2}]$

$P[T > 1] = e^{-2} = 0.1353$

(ii) $P[1 < T < 2] = \int_1^{2} f(t)\, dt$

$$= \int_1^2 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^2 = -[e^{-4} - e^{-2}]$$

$$= -[0.0183 - 0.1353]$$

$$= 0.117$$

(iii)$P[T \le 4] = \int_0^4 f(t)\, dt$

$$= \int_0^4 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_0^4 = -[e^{-8} - e^{-0}]$$

$$= -[3.35 * 10^{-4} - 1]$$

$= 0.996$

$P[T \leq 4] = 0.996$