

### 4.5. IIR FILTERS DESIGN -IMPULSE INVARIANT TRANSFORMATION

The objective of impulse invariant transformation is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. The main idea behind this technique is to preserve the frequency response characteristics of the analog filter. It can be stated that the frequency response of digital filter will be identical with the frequency response of the corresponding analog filter if the sampling time period T is selected sufficiently small to minimize the effect of aliasing.

Let  $h(t)$  = Impulse response of analog filter

Transfer function of analog filter,  $H(s) = L\{ h(t) \}$

$$H(s) = \sum_{i=1}^N \frac{A_i}{s+p_i} = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \dots + \frac{A_N}{s+p_N}$$

Let  $h(n)$  = Impulse response of the digital filter

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$$h(n) = h(t) |_{t=nT}$$

$$h(n) = h(t) |_{t=nT} = h(nT) = \sum_{i=1}^N A_i e^{-p_i nT} u(nT)$$

$$= A_1 e^{-p_1 nT} u(nT) + A_2 e^{-p_2 nT} u(nT) + A_3 e^{-p_3 nT} u(nT) + \dots + A_N e^{-p_N nT} u(nT)$$

On taking the Z transform of equation we get

$$H(z) = Z\{ h(n) \} = A_1 \frac{1}{1-e^{-p_1 T} z^{-1}} + A_2 \frac{1}{1-e^{-p_2 T} z^{-1}} + \dots + A_N \frac{1}{1-e^{-p_N T} z^{-1}}$$

Comparing the expression of  $H(s)$  and  $H(z)$  we can say that

$$\frac{1}{s+p_i} \rightarrow \frac{1}{1-e^{-p_i T} z^{-1}} \dots \dots \dots (1)$$

By impulse invariant transformation, where T is the sampling period.

Relation between Analog and digital filter poles in Impulse Invariant transformation

The analog poles are given by the roots of the term  $(s + p_i)$ , for  $i= 1,2,3\dots N$ . The digital poles are given by the roots of the term  $(1 - e^{-p_i T} z^{-1})$ , for  $i=1,2,3\dots N$ . From equation (1) we can say that the analog poles at  $s=-p_i$  is transformed into a digital pole at  $z=e^{-p_i T}$

The following observations can be made

1. If the analog pole  $s_i$  lie on Left Half of s-plane the corresponding digital pole  $z_i$  lie inside the unit circle in z plane.
2. If the analog pole  $s_i$  lie on Imaginary axis of s-plane the corresponding digital pole  $z_i$  lie on the unit circle in z plane.
3. If the analog pole  $s_i$  lie on the Right Half of s-plane the corresponding digital pole  $z_i$  lie on the outside the unit circle in z plane.

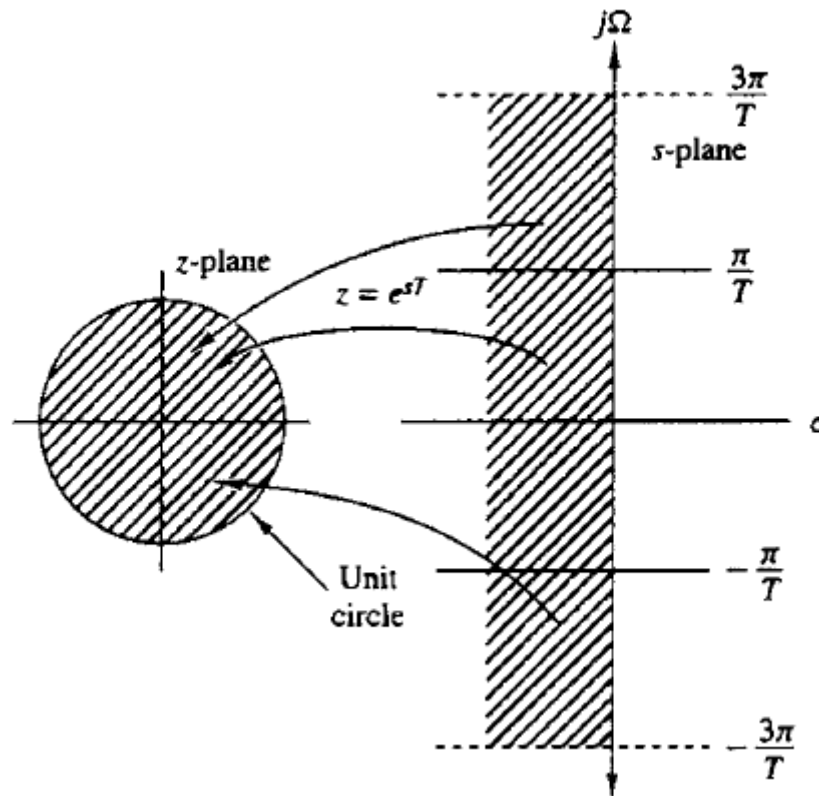


Fig.4.5.1. The mapping of s plane into z plane

[Source: 'Digital Signal Processing Principles, Algorithms and Applications' by J.G. Proakis and D.G. Manolakis page-674]

The stability of the filter is related to the location of the pole.

### Example -1

For the analog transfer function,  $H(s) = \frac{2}{s^2 + 3s + 2}$ , determine  $H(z)$  using impulse invariant transformation if (a)  $T=1$  second and (b)  $T=0.1$  second

Given that:

$$H(s) = \frac{2}{s^2 + 3s + 2}$$

$$= \frac{2}{(s+1)(s+2)}$$

By partial fraction expansion technique we can write

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \frac{2}{(s+1)(s+2)} \times (s+1) \Big|_{s=-1}$$

$$= \frac{2}{1} = 2$$

$$B = \frac{2}{(s+1)(s+2)} \times (s+2) \Big|_{s=-2} = \frac{2}{-2+1} = -2$$

$$H(s) = \frac{2}{s+1} + \frac{-2}{s+2}$$

$$= \frac{2}{s+1} - \frac{2}{s+2}$$

By impulse invariant transformation we know that

$$\frac{1}{s+p_i} \rightarrow \frac{1}{1-e^{-p_i T} z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-p_1 T} z^{-1}} + \frac{-2}{1-e^{-p_2 T} z^{-1}} \text{ where } p_1 = 1 \text{ and } p_2 = 2$$

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} + \frac{-2}{1-e^{-2T} z^{-1}}$$

a) When  $T=1$  second

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} + \frac{-2}{1-e^{-2T} z^{-1}}$$

$$H(z) = \frac{2}{1-0.3679 z^{-1}} + \frac{-2}{1-0.1353 z^{-1}}$$

$$= \frac{2}{1-0.3679 z^{-1}} - \frac{2}{1-0.1353 z^{-1}}$$

$$= \frac{2(1-0.1353 z^{-1}) - 2(1-0.3679 z^{-1})}{(1-0.3679 z^{-1})(1-0.1353 z^{-1})}$$

$$= \frac{2 - 0.2706 z^{-1} - 2 + 0.7358 z^{-1}}{1 - 0.1353 z^{-1} - 0.367 z^{-1} + 0.0498 z^{-2}}$$

$$= \frac{2 - 0.2706 z^{-1} - 2 + 0.7358 z^{-1}}{1 - 0.1353 z^{-1} - 0.367 z^{-1} + 0.0498 z^{-2}}$$

$$= \frac{0.4652 z^{-1}}{1 - 0.503 z^{-1} + 0.0498 z^{-2}}$$

$$\mathbf{H(z) = \frac{0.4652 z^{-1}}{1 - 0.503 z^{-1} + 0.0498 z^{-2}}}$$

b) When  $T=0.1$  second

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} + \frac{-2}{1-e^{-2T} z^{-1}}$$

$$= \frac{2}{1-e^{-0.1} z^{-1}} + \frac{-2}{1-e^{-0.2} z^{-1}}$$

$$\begin{aligned}
&= \frac{2}{1-0.9048z^{-1}} + \frac{-2}{1-0.8187z^{-1}} \\
&= \frac{(1-0.8187z^{-1})2 - 2(1-0.9048z^{-1})}{(1-0.9048z^{-1})(1-0.8187z^{-1})} \\
&= \frac{2 - 1.637z^{-1} - 2 + 1.8096z^{-1}}{1 - 0.8187z^{-1} - 0.9048z^{-1} + 0.7408z^{-2}} \\
&= \frac{0.1722z^{-1}}{1 - 1.7235z^{-1} + 0.7408z^{-2}}
\end{aligned}$$

$$\mathbf{H(z)} = \frac{0.1722z^{-1}}{1 - 1.7235z^{-1} + 0.7408z^{-2}}$$

