

Plane waves in Lossless medium

In a lossless medium, ϵ and μ are real numbers, so k is real.

In Cartesian coordinates each of the equations 6.1(a) and 6.1(b) are equivalent to three scalar Helmholtz's equations, one each in the components E_x , E_y and E_z or H_x , H_y , H_z .

For example if we consider E_x component we can write

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \dots\dots\dots(6.2)$$

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wavefront or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios.

Let us consider a plane wave which has only E_x component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z i.e., $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$

plane, . The Helmholtz's equation (6.2) reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \quad \dots\dots\dots(6.3)$$

The solution to this equation can be written as

$$\begin{aligned} E_x(z) &= E_x^+(z) + E_x^-(z) \\ &= E_0^+ e^{-jkz} + E_0^- e^{jkz} \quad \dots\dots\dots(6.4) \end{aligned}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions). In the time domain, $\epsilon_x(z, t) = \text{Re}(E_x(z)e^{j\omega t})$

$$\epsilon_x(z, t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz) \quad \dots\dots\dots(6.5)$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_x^+(z, t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. The $\varepsilon_x^+(z, t)$ plot for several values of t is shown in the Figure 2.1.

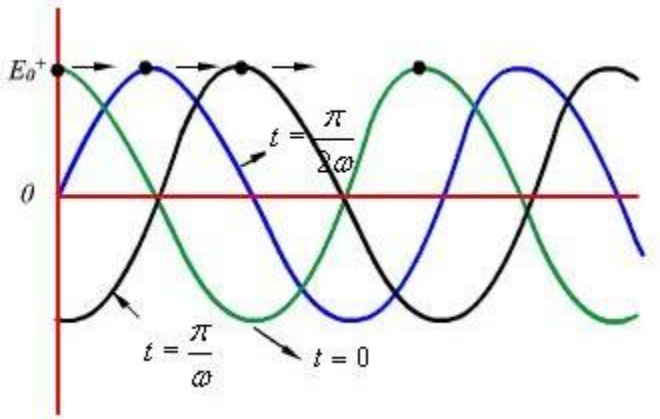


Figure 2.1: Plane wave traveling in the + z direction
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

As can be seen from the figure, at successive times, the wave travels in the +z direction. If we fix our attention on a particular point or phase on the wave (as shown by the dot)
i.e., $\omega t - \beta z = \text{constant}$

Then we see that as t is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$$\omega(t + \Delta t) - \beta(z + \Delta z) = \text{constant} = \omega t - \beta z$$

$$\text{Or,} \quad \omega \Delta t = \beta \Delta z$$

$$\text{Or,} \quad \frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}$$

When $\Delta t \rightarrow 0$,

$$\text{we write} \quad \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt} = \text{phase velocity } v_p$$

$$\therefore v_p = \frac{\omega}{\beta} \quad \text{.....(6.6)}$$

If the medium in which the wave is propagating is free space i.e.,

$$\epsilon = \epsilon_0, \mu = \mu_0$$

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = C$$

Then

Where 'C' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

$$\text{i.e., } (\omega t - kz) - [\omega t - k(z + \lambda)] = 2\pi$$

$$\text{or, } k\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{k}$$

$$\text{or,}$$

$$\text{Substituting } k = \frac{\omega}{v_p},$$

$$\lambda = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

$$\text{or, } \lambda f = v_p \dots \dots \dots (6.7)$$

Thus wavelength λ also represents the distance covered in one oscillation of the wave. Similarly, $E^-(z, t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as

follows: From (6.4),

$$\vec{E}_x^+(z) = E_0^+ e^{-jkz} \hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu} \nabla \times \vec{E}$$

$$= -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu} E_0^+ e^{-jkz} \hat{a}_y$$

$$= \dots\dots\dots(6.8)$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic impedance

of the medium. When the wave travels in free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi = 377\Omega \text{ of the free space.}$$

In the time domain,

$$\vec{H}^+(z, t) = \hat{a}_y \frac{E_0^+}{\eta} \cos(\omega t - \beta z) \dots\dots\dots(6.9)$$

Which represents the magnetic field of the wave traveling in the +z direction. For the negative traveling wave,

$$\vec{H}^-(z, t) = -\hat{a}_y \frac{E_0^+}{\eta} \cos(\omega t + \beta z) \dots\dots\dots(6.10)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The E & H field components of a TEM wave is shown in Fig 2.2.

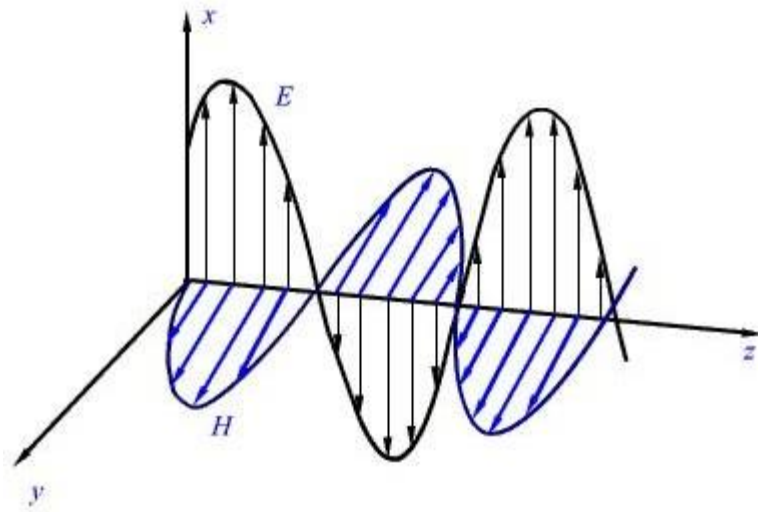


Figure 2.2 : E & H fields of a particular plane wave at time t.
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