

1.2 SUBSPACES

Definition :

Let V be a vector space and U be a non-empty subset of V . If U is a vector space under the operation of addition and scalar multiplication of V , then it is said to be a subspace of V .

Note:

- (i) $\{0\}$ and V itself are called trivial subspaces.
- (ii) All other vector subspace of V are called non-trivial subspaces.

Note :

- (i) A non-empty subset U of a vector space V over F is called subspace of V , if $u + v \in U$ and $\alpha u \in U$ for all $u, v \in U$ and $\alpha \in F$ or simply $\alpha u + \beta v \in U$ and $\alpha, \beta \in F$
- (ii) $\{0\}$ is a subspace of V called zero subspace.
- (iii) V is a subspace of its own.
- (iv) $\{0\}$ and V are called trivial subspace (or) improper subspaces.
- (v) Any subspace other than $\{0\}$ and V are called proper subspaces of V (or) non-trivial subspaces.
- (vi) The vectors lying on a line L through the origin \mathbb{R}^2 are subspaces of the vector space.

- (vii) A non-empty subset U of vector space V is a subspace iff $u + \alpha v \in U$ for any $v \in U$ and $\alpha \in F$.

Theorem : 1.

Let w_1 and w_2 be two subspaces of vector space V over F . Then $w_1 \cap w_2$ is a subspace of V .

Proof :

As $0 \in w_1 \cap w_2$, $w_1 \cap w_2$ is non-empty.

Consider $u, v \in w_1 \cap w_2, \alpha \in F$.

Then $u, v \in w_1, \alpha \in F$ and $u, v \in w_2, \alpha \in F$

$u + \alpha v \in w_1$ and $u + \alpha v \in w_2$

So, $u + \alpha v \in w_1 \cap w_2$

Hence $w_1 \cap w_2$ is a subspace of V .

PROBLEMS BASED ON SUBSPACES

- Let $V = \mathbb{R}^3$. The XY-plane $w_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and the XZ-plane $w_2 = \{(x, 0, z) : x, z \in \mathbb{R}\}$. These are subspaces of \mathbb{R}^3 . Then $w_1 \cap w_2 = \{(x, 0, 0) : x \in \mathbb{R}\}$ is the x-axis.

Solution :

Let $v \in V, v = (x, y, z) \in V$

$$v = (x, y, 0) + (0, 0, z) \in w_1 + w_2$$

$$\text{So, } V \subseteq w_1 + w_2 \subseteq V$$

$$\text{Hence } V = w_1 + w_2$$

2. Express the polynomial $3t^2 + 5t - 5$ as a linear combination of the polynomials $t^2 + 2t + 1, 2t^2 + 5t + 4, t^2 + 3t + 6$

Solution :

Let $a, b, c \in F$ such that

$$3t^2 + 5t - 5 = a(t^2 + 2t + 1) + b(2t^2 + 5t + 4) + c(t^2 + 3t + 6)$$

$$3t^2 + 5t - 5 = (a + 2b + c)t^2 + (2a + 5b + 3c)t + (a + 4b + 6c)$$

Comparing the co-efficients, we get

$$a + 2b + c = 3 \quad \dots(1)$$

$$2a + 5b + 3c = 5 \quad \dots(2)$$

$$a + 4b + 6c = -5 \quad \dots(3)$$

$$(3) - (1) \Rightarrow \quad 2b + 5c = -8 \quad \dots(4)$$

Multiply (1) by 2,

$$2a + 4b + 2c = 6 \quad \dots\dots\dots(5)$$

$$(2) - (5) \Rightarrow b + c = -1 \dots (6)$$

Multiply (6) by 2,

$$2b + 2c = -2 \dots (7)$$

$$(4) - (7) \Rightarrow 3c = -6$$

$$\therefore c = -2$$

Substituting c in (6),

$$b - 2 = -1$$

$$b = 2 - 1 = 1$$

$$\therefore b = 1$$

Substituting c, b in (1)

$$a + 2(1) - 2 = 3$$

$$a + 2 - 2 = 3$$

$$\Rightarrow a = 3$$

$$\therefore a = 3, b = 1, c = -2$$

$$\text{Hence, } 3t^2 + 5t - 5 = 3(t^2 + 2t + 1) + 1(2t^2 + 5t + 4)$$

$$-2(t^2 + 3t + 6)$$

3. Let $V = R^3$, then which of the following sets is/are subspace(s) of V .

$$(i) w_1 = \{(a, b, 0); a, b \in \mathbb{R}\}$$

$$(ii) w_2 = \{(a, b, 0); a \geq 0\}$$

Solution :

$$(i) \quad \bar{0} = (0, 0, 0) \in w_1, \text{ so } w_1 \neq \phi$$

Let $v_1, v_2 \in w_1, \alpha \in \mathbb{R}$

Then, $v_1 = (a, b, 0)$ and $v_2 = (c, d, 0)$ for some $a, b, c, d \in \mathbb{R}$

$$v_1 + v_2 = (a + c, b + d, 0) \in w_1$$

$$\alpha v_1 = (\alpha a, \alpha b, 0) \in w_1$$

Hence w_1 is a subspace of V .

$$(ii) \text{ Consider } w_2 = \{(a, b, 0); a \geq 0\}$$

Here we should take the value of a as zero or positive.

$$\text{Let } V = (2, 1, 0) \in w_2$$

But under scalar multiplication, the vector is not in w_2

$$\text{That is } -v = (-2, -1, 0) \notin w_2$$

$$(-1)v \notin w_2$$

Hence w_2 is not a subspace of V

4. Let V be a vector space of all 2×2 matrices over real numbers. Determine whether W is a subspace of V or not, where

(i) W consists of all matrices with non-zero determinant.

(ii) W consists of all matrices A such that $A^2 = A$.

Solution :

$$(i) \text{ Let } W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$, W is a non-empty subset of V .

Consider $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \in W$ and $\alpha, \beta \in \mathbb{R}$

$$\alpha A = \begin{bmatrix} \alpha x_1 & 0 \\ 0 & \alpha y_1 \end{bmatrix} \text{ and } \alpha B = \begin{bmatrix} \beta x_2 & 0 \\ 0 & \beta y_2 \end{bmatrix}$$

$$\alpha A + \beta B = \begin{bmatrix} \alpha x_1 + \beta x_2 & 0 \\ 0 & \alpha y_1 + \beta y_2 \end{bmatrix} \in W$$

Hence W is a subspace of V .

(ii) W is not a subspace of V because W is not closed under addition.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

$$\therefore A \in W$$

$$\text{But } A + A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq A + A$$

Thus $A + A \notin W$

7. Let $V = \{A/A = [a_{ij}]_{n \times n}, a_{ij} \in \mathbf{R}\}$ be a vector space over \mathbf{R} . Show $W = \{A \in V/AX = XA \text{ for all } X \in V\}$ is a sub-space of $V(\mathbf{R})$

Solution :

Since $0X = 0 = X0$ for all $X \in V$

$\Rightarrow 0 \in W$. Thus W is non-empty.

Now, let $\alpha, \beta \in \mathbf{R}$ and $A_1, A_2 \in W$

$\Rightarrow A_1X = XA_1$ and $A_2X = XA_2$ for all $X \in V$

$$\begin{aligned} \therefore (\alpha A_1 + \beta A_2)X &= (\alpha A_1)X + (\beta A_2)X \\ &= \alpha(A_1X) + \beta(A_2X) \end{aligned}$$

$$= \alpha(XA_1) + \beta(XA_2)$$

$$= X(\alpha A_1) + X(\beta A_2)$$

$$= X(\alpha A_1 + \beta A_2)$$

$$= \alpha A_1 + \beta A_2 \in W$$

Hence W is a vector space of $V(\mathbf{R})$.

Theorem : 3. If S is any subset of a vector space $V(F)$, then S is a subspace of $V(F)$ if and only if $L(S) = S$.

Proof:

Given S is a subspace of $V(F)$

To prove $L(S) = S$

Let $x \in L(S) \Rightarrow$ there exists $x_1, \dots, x_n \in S$

$\alpha_1, \alpha_2, \dots, \alpha_n \in F$

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \in S$$

$$L(S) \subset S \quad \dots (1)$$

Also $S \subset L(S) \dots (2)$ [Since S is a subspace of $V(F)$]

From (1) and (2), $L(S) = S$

Conversely, Given $L(S) = S$

To prove: S is a subspace of $V(F)$

Since $L(S)$ is a subspace of $V(F)$

$\therefore S$ is also a subspace of $V(F)$

8. Let V be the set of all solutions of the differential equation $2y'' - 7y' + 3y = 0$. Then V is a vector space over R .

Solution :

Let $f, g \in V$ and $\alpha \in R$.

Then $2f'' - 7f' + 3f = 0$ and

$$2g'' - 7g' + 3g = 0$$

$$2 \frac{d^2}{dx^2} (f + g) - 7 \frac{d}{dx} (f + g) + 3(f + g) = 0$$

Hence $f + g \in V$

$$\text{Also } 2(\alpha f)'' - 7(\alpha f)' + 3(\alpha f) = 0$$

Hence $\alpha f \in V$

Hence V is a vector space over R .

9 Examine whether $(1, -3, 5)$ belongs to the linear space generated by S , where $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$ or not?

Solution :

Suppose $(1, -3, 5)$ belongs to S .

\therefore There exists scalars α, β, γ such that

$$(1, -3, 5) = \alpha(1, 2, 1) + \beta(1, 1, -1) + \gamma(4, 5, -2)$$

$$(1, -3, 5) = (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma)$$

Comparing both sides, we get

$$\alpha + \beta + 4\gamma = 1 \quad \dots\dots\dots(1)$$

$$2\alpha + \beta + 5\gamma = -3 \quad \dots\dots\dots(2)$$

$$\alpha - \beta - 2\gamma = 5 \quad \dots\dots\dots(3)$$

Adding (1) and (3), we get

$$2\alpha + 2\gamma = 6 \Rightarrow \alpha + \gamma = 3 \quad \dots \quad (4)$$

Adding (2) and (3), we get

$$3\alpha + 3\gamma = 2 \Rightarrow \alpha + \gamma = \frac{2}{3} \quad \dots \quad (5)$$

Equation (4) and (5) are contradiction

Hence $(1, -3, 5)$ does not belong to linear space of S .

Remark :

The union of the subspace may not be a sub-space.