Double integrals in Polar co-ordinates

Consider the integral

$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)dr\,d\theta$$

which is in polar form. This integral is bounded over the region by the straight line $\theta = \theta_1, \theta = \theta_2$ and the curves $r = r_1, r = r_2$.

Example:

Evaluate $\int_0^{\pi}\int_0^{sin\theta} rdrd\theta$

Solution:

Given $\int_0^{\pi}\int_0^{sin\theta} rdrd\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta$$

$$= \int_0^{\pi}\frac{sin^2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{1-cos2\theta}{2}\right] d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{sin2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{\pi}{4}$$

Example:

Evaluate $\int_0^{\pi}\int_0^{a} rdrd\theta$

Solution:

Given $\int_0^{\pi}\int_0^{a} rdrd\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a} d\theta$$

$$= \int_0^{\pi}\frac{a^2}{2} d\theta$$

$$= \frac{a^2}{2}[\theta]_0^{\pi}$$

$$= \frac{\pi a^2}{2}$$

Example:

Evaluate $\int_{-\pi/2}^{\pi/2}\int_0^{2cos\theta} r^2 d\theta dr$

Solution:

Given $\int_{-\pi/2}^{\pi/2}\int_0^{2cos\theta} r^2 d\theta dr$

$=\int_{-\pi/2}^{\pi/2}\int_{0}^{2cos\theta} r^2 drd\theta$ (correct form)

$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2cos\theta} d\theta$

$=\int_{-\pi/2}^{\pi/2}\left[\frac{(2cos\theta)^3}{3}-0\right]d\theta$

$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} cos^3\theta\, d\theta$

$=\frac{8}{3}(2)\int_{0}^{\pi/2} cos^3\theta\, d\theta$

$=\frac{16}{3}\left[\frac{2}{3}.1\right] = \frac{32}{9}$

Example:

Evaluate $\int_0^{\pi/2}\int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

Solution:

Given $\int_0^{\pi/2}\int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-cos\theta)}^{a} d\theta$

$= \int_0^{\pi/2}\left[\frac{a^3}{3}-\frac{a^3(1-cos\theta)^3}{3}\right]d\theta$

$= \frac{a^3}{3}\int_0^{\pi/2}[1-(1-cos\theta)^3]\mathrm{d}\theta$

$= \frac{a^3}{3}\int_0^{\pi/2}[1-(1-3cos\theta+3cos^2\theta-cos^3\theta)]\mathrm{d}\theta$

$= \frac{a^3}{3}\int_0^{\pi/2}[3cos\theta+3cos^2\theta-cos^3\theta)]d\theta$

$=\frac{a^3}{3}\left[(3sin\theta)_0^{\pi/2} - 3\frac{1}{2}\frac{\pi}{2}+\frac{2}{3}\right]$

$=\frac{a^3}{3}\left[3-3\frac{\pi}{2}+\frac{2}{3}\right]$

$=\frac{a^3}{3}\left[\frac{36-9\pi+8}{12}\right]$

$=\frac{a^3}{36}[44-9\pi]$