NATURE OF QUADRATIC FORM DETERMINED BY PRINCIPAL MINORS

Let A be a square matrix of order n say $A = \left[\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ddots & a_{2n} \\ & \dots & & \cdots & \dots. \end{pmatrix}\right]$

The principal sub determinants of A are defined as below.

$$s_1 = a_{11}$$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$s_n = |A|$$

The quadratic form $Q = X^T AX$ is said to be

1. Positive definite: If $s_{1,}s_{2,}s_{3\dots\dots\dots}s_n > 0$
2. Positive semidefinite: If $s_{1,}s_{2,}s_{3\dots\dots\dots}s_n \geq 0$ and atleast one $s_i = 0$
3. Negative definite: If $s_{1,}s_{3,}s_{5\dots\dots\dots} < 0$ and $s_{2,}s_{4,}s_{6\dots\dots\dots} > 0$
4. Negative semidefinite: If $s_{1,}s_{3,}s_{5\dots\dots\dots} < 0$ and $s_{2,}s_{4,}s_{6\dots\dots\dots} > 0$ and atleast one $s_i = 0$
5. Indefinite: In all other cases

Example: Determine the nature of the Quadratic form $\mathbf{12x_1^2 + 3x_2^2 + 12x_3^2 + 2x_1x_2}$

Solution:

$$A = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$s_1 = a_{11} = 12 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & 1 \\ 1 & 3 \end{vmatrix} = 35 > 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{vmatrix} = 430 > 0$$, Postive definite

Example: Determine the nature of the Quadratic form $\mathbf{x_1^2 + 2x_2^2}$

Solution:

Let A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$s_1 = a_{11} = 1 > 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$, Positive semidefinite

Example: Determine the nature of the Quadratic form $\mathbf{x^2 - y^2 + 4z^2 + 4xy + 2yz + 6zx}$

Solution:

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$

$s_1 = a_{11} = 1 > 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -5 < 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0$, Indefinite

Example: Determine the nature of the Quadratic form $\mathbf{xy + yz + zx}$

Solution:

Let A = $\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

$s_1 = a_{11} = 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -1/4 < 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{vmatrix} = \frac{1}{4} > 0$, Indefinite

RANK, INDEX AND SIGNATURE OF A REAL QUADRATIC FORMS

Let $Q = X^T AX$ be quadratic form and the corresponding canonical form is $d_1{y_1}^2 + d_2{y_2}^2 + \cdots + d_n{y_n}^2$.

The **rank** of the matrix A is number of non –zero Eigen values of A. If the rank of A is ‘r’, the canonical form of Q will contain only “r” terms .Some terms in the canonical form may be positive or zero or negative.

The number of positive terms in the canonical form is called the **index**(p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form .i.e,$p-(r-p)=2p-r$ is called the signature of the quadratic form and usually denoted by s. Thus $s=2p-r$.