TRANSFORM OF PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T)=f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T)=f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t+2\pi) = \sin(t+4\pi)\cdots$

$\therefore sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

Example: Find the Laplace transform of $f(t)=\begin{cases} sin\omega t; 0<t<\frac{\pi}{\omega} \\ 0; \frac{\pi}{\omega}<t<\frac{2\pi}{\omega}\end{cases}$ $f\left(t+\frac{2\pi}{\omega}\right)=f(t)$

Solution:

The given function is a periodic function with period $T=\frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st}dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}(0)dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st}dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-ssin\omega t-\omega cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left\{\frac{e^{\frac{-s\pi}{\omega}}}{s^2+\omega^2}[-ssin\pi-\omega cos\pi]+\frac{\omega}{s^2+\omega^2}\right\}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-\pi s}{\omega}}\right)^2}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

Example: Find the Laplace transform of $\boldsymbol{f(t)=\begin{cases} E;0\le t\le a \\ -E\,;a\le t\le 2a\end{cases}}$ **given that** $\boldsymbol{f(t+2a)=f(t)}$.

Solution:

The given function is a periodic function with period $T=2a$

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt+\int_a^{2a}-Ee^{-st}dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left[E\int_0^a e^{-st}dt-E\int_a^{2a} e^{-st}dt\right]$$

$$=\frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a-\left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$=\frac{E}{1-e^{-2as}}\left[\frac{e^{-as}}{-s}+\frac{1}{s}-\frac{e^{-2as}}{s}-\frac{e^{-as}}{s}\right]$$

$$=\frac{E}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$

$$=\frac{E}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$=\frac{E}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$=\frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)]=\frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: Find the Laplace transform of $\boldsymbol{f(t)=\begin{cases} 1;0\le t\le \frac{a}{2} \\ -1\,;\frac{a}{2}\le t\le a\end{cases}}$ **given that** $\boldsymbol{f(t+a)=f(t)}$.

Solution:

The given function is a periodic function with period $T=a$

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}}(1)e^{-st}dt+\int_{\frac{a}{2}}^a(-1)e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}}e^{-st}dt-\int_{\frac{a}{2}}^a e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}}-\left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{2a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because tanhx = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Example: Find the Laplace transform of $\boldsymbol{f(t) = \begin{cases} t; 0 \le t \le a \\ 2a - t\,; a \le t \le 2a \end{cases}}$ **given that** $\boldsymbol{f(t+2a) = f(t)}$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^a - \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$