Double Integrals

Let $f(x, y)$ be a single valued function and continuous in a region R bounded by a closed curve C. Let the region R be subdivided in any manner into n sub regions $R_1, R_2, R_3, \cdots, R_n$ of areas $A_1, A_2, A_3, \cdots, A_n$.Let (x_i, y_j) be any point in the sub regionR_i. Then consider the sum formed by multiplying the area of each sub – region by the value of the function $f(x, y)$ at any point of the sub – region and adding up the products which we denote

$$\sum_1^n f(x_i, y_j) A_i$$

The limit of this sum (if it exists) as $n \to \infty$ in such a way that each $A_i \to 0$ is defined as the double integral of $f(x, y)$ over the region R. Thus

$$\lim_{n \to \infty} \sum_1^n f(x_i, y_j) A_i = \iint_R f(x, y)\, dA$$

The above integral can be given as

$$\iint_R f(x, y) dydx \quad or \quad \iint_R f(x, y) dxdy$$

Evaluation of Double Integrals

To evaluate $\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) dx\, dy$ we first integrate f(x, y) with respect to x partially, that is treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be function of y. Then we integrate this function of with respect to y between the limits y_0 and $y_1$as used.

Example:

Evaluate $\int_0^1 \int_1^2 x(x + y) dydx$

Solution:

$$\int_0^1 \int_1^2 x(x + y) dydx = \int_0^1 \int_1^2 (x^2 + xy) dydx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \left[(2x^2 + 2x) - (x^2 + \frac{x}{2})\right] dx$$

$$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2})\right] dx$$

$$= \int_0^1 \left[x^2 + \frac{3}{2} x\right] dx$$

$$= \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2}\right]_0^1 \quad = \left(\frac{1}{3} + \frac{3}{4}\right) - (0 + 0) \quad = \frac{13}{12}$$

Example:

Evaluate $\int_0^a \int_0^b xy(x - y) dydx$

Solution:

$$\int_0^a \int_0^b xy(x-y)dydx \quad = \int_0^a \int_0^b (x^2y - xy^2)dydx$$

$$= \int_0^a \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^b dx$$

$$= \int_0^a \left[\left(\frac{b^2x^2}{2} - \frac{b^3x}{2}\right) - (0-0)\right] dx$$

$$= \left[\left(\frac{b^2x^3}{6} - \frac{b^3x^2}{6}\right)\right]_0^a$$

$$= \left(\frac{a^3b^2}{6} - \frac{a^2b^3}{6}\right) - (0-0)$$

$$= \frac{a^2b^2}{6}(a-b)$$

Example:

Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

Solution:

$$\int_2^a \int_2^b \frac{dxdy}{xy} \quad = \int_2^a \left[\frac{1}{y} logx\right]_2^b dy$$

$$= \int_2^a \frac{1}{y}(logb - log2)dy$$

$$= \int_2^a \frac{1}{y} log\left(\frac{b}{2}\right) dy \qquad \left[\because log\frac{a}{b} = loga - logb\right]$$

$$= log\frac{b}{2}\int_2^a \frac{1}{y}dy \quad = log\frac{b}{2}[\log y]_2^a$$

$$= log\frac{b}{2}[loga - log2] \quad = \left[log\frac{b}{2}\right]\left[log\frac{a}{2}\right]$$

Example:

Evaluate $\int_0^1 \int_2^3 (x^2 + y^2)dxdy$

Solution:

$$\int_0^1 \int_2^3 (x^2+y^2)dxdy \quad = \int_0^1 \left[\frac{x^3}{3} + y^2x\right]_2^3 dy$$

$$= \int_0^1 \left[\left(\frac{3^3}{3} + 3y^2\right) - \left(\frac{2^3}{3} + 2y^2\right)\right] dy$$

$$= \int_0^1 \left[9 + 3y^2 - \frac{8}{3} - 2y^2\right] dy$$

$$= \int_0^1 \left[\frac{19}{3} + y^2\right] dy \quad = \left[\frac{19y}{3} + \frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{19}{3} + \frac{1}{3}\right] = \frac{20}{3}$$

Example:

Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dydx$

Solution:

$$\int_0^3 \int_0^2 e^{x+y}\, dydx \;= \int_0^3 \int_0^2 e^x\, e^y dydx \;= \left[\int_0^3 e^x\, dx\right]\left[\int_0^2 e^y\, dy\right]$$

$$= [e^x]_0^3 [e^y]_0^2 \;= [e^3 - e^0][e^2 - e^0]$$

$$= [e^3 - 1][e^2 - 1]$$

Note: If the limits are variable, then check the given problem is in the correct form

Rule: (i) The limits for the inner integral are functions of , then the first integral is with respect to y

(ii) The limits for the inner integral are functions of , then the first integral is with respect to x

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx \;= \int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx \;= \int_0^a \left[\sqrt{a^2-x^2}\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\left(0 + \frac{a^2}{2} sin^{-1}1\right) - (0+0)\right] \qquad \left[\because sin^{-1}1 = \frac{\pi}{2}, sin^{-1}0 = 0\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) \;= \frac{\pi a^2}{4}$$

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dydx \;= \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y+y^3)dydx$$

$$= \int_0^a \left[\frac{x^2y^2}{2} + \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{x^2(a^2-x^2)}{2} + \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \int_0^a \left[\frac{a^2x^2}{2} - \frac{x^4}{2} + \frac{a^4}{4} + \frac{x^4}{4} - \frac{2a^2x^2}{4}\right] dx$$

$$=\left[\frac{a^2x^3}{6}-\frac{x^5}{10}+\frac{a^4x}{4}+\frac{x^5}{20}-\frac{2a^2x^3}{12}\right]_0^a$$

$$=\left[\frac{-x^5}{10}+\frac{a^4x}{4}+\frac{x^5}{20}\right]_0^a$$

$$=\left[\frac{-a^5}{10}+\frac{a^5}{4}+\frac{a^5}{20}\right]$$

$$=\frac{a^5}{5}$$

Example:

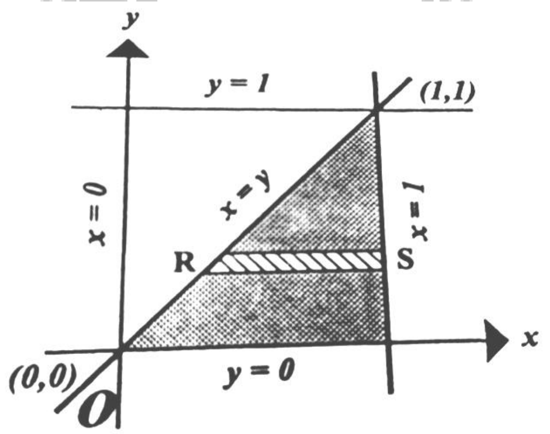
Sketch roughly the region of integration for $\int_0^1\int_0^x f(x,y)\,dydx$

Solution:

Given $\int_0^1\int_0^x f(x,y)\,dydx$

x varies from $x=0$ to $x=1$

y varies from $y=0$ to $y=x$



Example:

Shade the region of integration $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$

Solution:

$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dydx$ is the correct form

x limit varies from $x=0$ to $x=a$

y limit varies from $y=\sqrt{ax-x^2}$ to $y=\sqrt{a^2-x^2}$

i.e., $y^2=ax-x^2$ to $y^2=a^2-x^2$

i.e., $y^2+x^2=ax$ to $y^2+x^2=a^2$

$x^2+y^2=ax$ is a circle with centre $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$

$x^2 + y^2 = a^2$ is a circle with centre $(0,0)$ and radius a