2.4 Recurrence Relations:

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non – negative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a sequence satisfies a recurrence relation, then the sequence is called a solution of the recurrence relation.

For example, we consider the famous Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, . . .

Which can be represented by the recurrence relation.

$$F_n = F_{n-1} + F_{n-2}, n \geq 2$$

and $F_0 = 0, F_1 = 1$

Here, $F_0 = 0, F_1 = 1$ are called initial conditions.

It is a second order recurrence relation.

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_K k a_{n-k}$$

Where $C_1, C_2, \ldots, C_k$ are real numbers, and $C_k \neq 0$.

The recurrence relation in the definition is linear since the right – hand side is a sum of multiplies of the previous terms of the sequence.

The recurrence relation is homogeneous, since no terms occur that are not multiplies of the a_j's.

The coefficients of the terms of the sequence are all constants, rather than function that depend on "n".

The degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Solving Linear Homogeneous Recurrence Relations With Constant Coefficients:

Step: 1 Write down the characteristic equation for the given recurrence relation. Here, the degree of character equation is 1 less than the number of terms in recurrence relation.

Step: 2 By solving the characteristic equation find out the characteristic roots.

Step: 3 Depends upon the nature of roots, find out the solution a_n as follows:

Case (i) Let the roots be real and distinct say $r_1, r_2, \ldots, r_n$.

Then $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n + \alpha_3 r_3{}^n + \ldots + \alpha_n r_n{}^n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case (ii) Let the roots be real and equal say $r_1 = r_2 = \ldots = r_n$.

Then $a_n = \alpha_1 r_1{}^n + n\alpha_2 r_2{}^n + n^2 \alpha_3 r_3{}^n + \ldots + n^n \alpha_n r_n{}^n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case (iii) When the roots are complex conjugate, then

$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$

Step: 4 Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation

$S(k) = 2[S(k-1)]^2 - kS(k-3)$ cannot be solved.

1. If the sequence $a_n = 3 \cdot 2^n, n \geq 1$, then find the corresponding recurrence relation.

Solution:

Given $a_n = 3 \cdot 2^n$

$\Rightarrow a_{n-1} = 3 \cdot 2^{n-1}$

$= 3 \cdot \frac{2^n}{2}$

$\Rightarrow a_{n-1} = \frac{a^n}{2}$

$\Rightarrow a_n = 2(a_{n-1})$

Hence $a_n = 2a_{n-1}, n \geq 1$ with $a_0 = 3$

2. Find the recurrence relation for $S(n) = 6(-5)^n, n \geq 0$

Solution:

Given $S(n) = 6(-5)^n$

$\Rightarrow S(n-1) = 6(-5)^{n-1}$

$= 6\frac{(-5)^n}{-5}$

$= \frac{S(n)}{-5}$

$\Rightarrow S(n) = -5 \cdot S(n-1), n \geq 0$ with $S(0) = 6$

3. Find the recurrence relation from $y_k = A \cdot 2^k + B \cdot 3^k$

Solution:

Given $y_k = A \cdot 2^k + B \cdot 3^k \quad \ldots (1)$

$\Rightarrow y_{k+1} = A \cdot 2^{k+1} + B \cdot 3^{k+1}$

$= A \cdot 2^k \cdot 2 + B \cdot 3^k \cdot 3$

$= 2A \cdot 2^k + 3B \cdot 3^k \qquad ...(2)$

$\Rightarrow y_{k+2} = 4A \cdot 2^k + 9B \cdot 3^k \qquad ...(3)$

$(3) - 5(2) + 6(1)$

$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 4A \cdot 2^k + 9B \cdot 3^k - 10A \cdot 2^k - 15B \cdot 3^k + 6A \cdot 2^k + 6B \cdot 3^k = 0$

$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 0$

4. Find the recurrence relation from $y_n = A3^n + B(-4)^n$

Solution:

Given $y_n = A3^n + B(-4)^n \qquad ...(1)$

$\Rightarrow y_{n+1} = y_n = A3^{n+1} + B(-4)^{n+1}$

$= A3^n \cdot 3 + B(-4)^n \cdot (-4)$

$= 3A \cdot 3^n - 4B \cdot (-4)^n \qquad ...(2)$

$\Rightarrow y_{n+2} = 9A \cdot 3^n + 16B \cdot (-4)^n \qquad ...(3)$

$(3) + (2) - 12(1)$

$\Rightarrow y_{n+2} + y_{n+1} - 12y_n = 9A3^n + 16B(-4)^n + 3A3^n - 4B(-4)^n - 12A3^n - 12B(-4)^n = 0$

$\Rightarrow y_{n+2} + y_{n+1} - y_n = 0$

5. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5, a_2 = 15$

Solution:

The recurrence relation can be written as $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$

By solving, we get the characteristic roots, $r = 1, 2, 3$

Solution is $a_n = \alpha_1 \cdot 1^n + \alpha_2 2^n + \alpha_3 3^n$ . . . (A)

Given $a_0 = 2$, Put $n = 0$ in (A)

$$a_0 = \alpha_1 \cdot (1)^0 + \alpha_2(2)^0 + \alpha_3(3)^0$$

(A) $\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2$ . . . (1)

Given $a_1 = 5$, Put $n = 1$ in (A)

$$a_1 = \alpha_1 \cdot (1)^1 + \alpha_2(2)^1 + \alpha_3(3)^1$$

(A) $\Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$ . . . (2)

Given $a_2 = 15$, Put $n = 2$ in (A)

$$a_2 = \alpha_1 \cdot (1)^2 + \alpha_2(2)^2 + \alpha_3(3)^2$$

(A) $\Rightarrow \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$ . . . (3)

To solve (1), (2) and (3)

(1) $\Rightarrow \alpha_3 = 2 - \alpha_1 - \alpha_2$ . . . (4)

Using (4) in (2)

$(2) \Rightarrow 2\alpha_1 + \alpha_2 = 1$ . . . (5)

Using (4) in (3)

$(3) \Rightarrow 8\alpha_1 + 5\alpha_3 = 3$ . . . (6)

Solving (5) and (6), we get $\alpha_1 = 1$ and $\alpha_2 = -1$

Using $\alpha_1 = 1$ and $\alpha_2 = -1$ in (1) we get $\alpha_3 = 2$

Substituting $\alpha_1 = 1$ and $\alpha_2 = -1$ and $\alpha_3 = 2$ in (A), we get

Solution is $a_n = 1 \cdot 1^n - 1 \cdot 2^n + 2 \cdot 3^n$